NONLINEAR HENCKY'S BEAM MODEL FOR BEAM STRUCTURES

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Outline

Outline

Pendulum-Based System Representation

- Horizontal multi-pendulums: thin rods are rotationally connected.
- System positions: generalized coordinates (ϕ_i) of the rods
- The continuous beam is discretized:
  1. The whole beam is divided into n equal segments;
  2. A joint with rotational spring and damper is placed in the middle of each one.
  3. The result: n+1 pendulums, numbered from 0 to n.
  4. The lengths of each: l_e, except for the 0th and n-th pendulums $l_e/2$.
- General boundary conditions: by attaching three springs: k_x, k_y, and k_ϕ.
- An additional longitudinal spring ($k_s t$) connected in series.

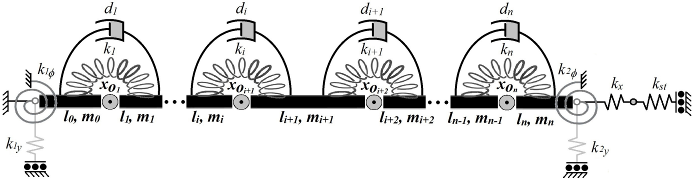


Figure 1: Nonlinear Henckey Model: Pendulum-Based System Representation.

Outline

Lagrange's equations of the second kind

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}_i}\right) - \frac{\partial T}{\partial \phi_i} + \frac{\partial V}{\partial \phi_i} + \frac{\partial R}{\partial \dot{\phi}_i} = Q_i, \quad i = 1 \ldots n \tag{1}$$

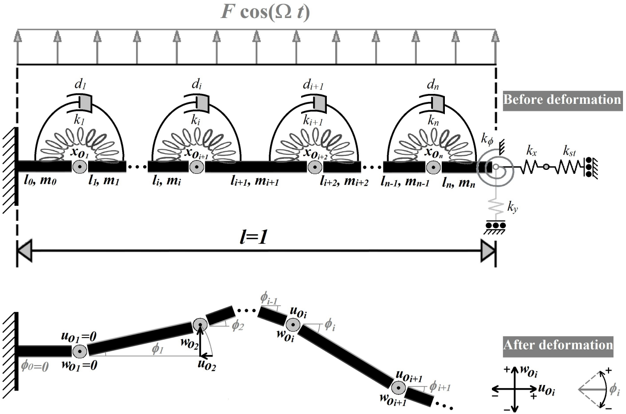


Figure 2: System of n pendulums representing the beam's segments.

System Energies $\left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}_i}\right) - \frac{\partial T}{\partial \phi_i} + \frac{\partial V}{\partial \phi_i} + \frac{\partial R}{\partial \dot{\phi}_i} = Q_i\right)$

- Kinetic energy T of the system:

$$T = \frac{1}{2}\sum_{i=1}^{n}\left[m_i\left(\dot{u}_{C_i}^2 + \dot{w}_{C_i}^2\right) + B_i\dot{\phi}_i^2\right] \tag{2}$$

- Potential energy V due to rotational springs and external stiffness:

$$V = \frac{1}{2}\sum_{i=1}^{n} k_i(\phi_i - \phi_{i-1})^2 + \frac{1}{2}k_x^{Total}u_{end}^2 + \frac{1}{2}k_y w_{end}^2 + \frac{1}{2}k_\phi \phi_n^2 \tag{3}$$

- The Rayleigh function R for internal damping:

$$R = \frac{1}{2}\sum_{i=1}^{n} d_i(\dot{\phi}_i - \dot{\phi}_{i-1})^2 \tag{4}$$

$$\left(T = \tfrac{1}{2}\sum_{i=1}^{n}\left[m_i\left(\dot{u}_{C_i}^2 + \dot{w}_{C_i}^2\right) + B_i\dot{\phi}_i^2\right], V = \tfrac{1}{2}\sum_{i=1}^{n} k_i(\phi_i - \phi_{i-1})^2 + \tfrac{1}{2}k_x^{Total}u_{end}^2 + \tfrac{1}{2}k_y w_{end}^2 + \tfrac{1}{2}k_\phi \phi_n^2\right)$$

- Nondimensional masses and mass moments of inertia:

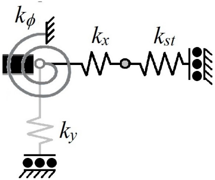
$$m_i = \frac{l_i}{l_e n}, \quad B_i = \frac{m_i (l_i)^2}{12}. \tag{5}$$

- The rotational stiffness of springs at each joint is:

$$k_i = n, \quad \phi_0 = 0.$$

- k_x is connected in series with k_{st}:

$$k_{st} = \frac{12 l^2}{t_b{}^2}, \quad k_x^{Total} = \frac{k_x k_{st}}{k_x + k_{st}}. \tag{6}$$



- External forces Q_i due to base excitation are:

$$Q_i = F\cos(\Omega t)\sum_{j=1}^{n} m_j \frac{\partial w_{C_j}}{\partial \phi_i} \tag{7}$$

Matrix Formulation

$$\mathbf{M} \cdot \ddot{\phi} + \mathbf{K} \cdot (\phi + d\,\dot{\phi}) + \mathbf{N} \cdot \dot{\phi}^2 + \mathbf{K}_B \cdot \phi = \mathbf{F} \tag{8}$$

To simplify the matrices for presentation purposes, we introduce indexing parameters:

$$s_i = \sin(\phi_i(t)), \qquad s_{i,j} = \sin(\phi_i(t) - \phi_j(t)), \tag{9a}$$

$$c_i = \cos(\phi_i(t)), \qquad c_{i,j} = \cos(\phi_i(t) - \phi_j(t)). \tag{9b}$$

$$sc_n = \frac{\sum_{j=1}^{n-1} c_j}{n^2} + \frac{c_n}{2n^2}, \qquad ss_n = \frac{\sum_{j=1}^{n-1} s_j}{n^2} + \frac{s_n}{2n^2}. \tag{9c}$$

Matrix Constructions ($\mathbf{M}\cdot\ddot{\phi}+\mathbf{K}\cdot(\phi+d\,\dot{\phi})+\mathbf{N}\cdot\dot{\phi}^2+\mathbf{K}_B\cdot\phi=\mathbf{F}$)

The matrices in Eq. (8) are constructed as follows:

1. **Mass matrix M**: This is an $n \times n$ symmetric matrix ($\mathbf{M}=\mathbf{M}^\top$):

$$\mathbf{M} = \begin{pmatrix} \frac{6n-7}{6n^3} & \frac{(n-1)c_{1,2}}{n^3} & \cdots & \frac{3c_{1,n-3}}{n^3} & \frac{2c_{1,n-2}}{n^3} & \frac{c_{1,n-1}}{n^3} & \frac{c_{1,n}}{8n^3} \\ \frac{(n-1)c_{1,2}}{n^3} & \frac{6n-13}{6n^3} & \cdots & \frac{3c_{2,n-3}}{n^3} & \frac{2c_{2,n-2}}{n^3} & \frac{c_{2,n-1}}{n^3} & \frac{c_{2,n}}{8n^3} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \frac{3c_{1,n-3}}{n^3} & \frac{3c_{2,n-3}}{n^3} & \cdots & \frac{17}{6n^3} & \frac{2c_{n-3,n-2}}{n^3} & \frac{c_{n-3,n-1}}{n^3} & \frac{c_{n-3,n}}{8n^3} \\ \frac{2c_{1,n-2}}{n^3} & \frac{2c_{2,n-2}}{n^3} & \cdots & \frac{2c_{n-3,n-2}}{n^3} & \frac{11}{6n^3} & \frac{c_{n-2,n-1}}{n^3} & \frac{c_{n-2,n}}{8n^3} \\ \frac{c_{1,n-1}}{n^3} & \frac{c_{2,n-1}}{n^3} & \cdots & \frac{c_{n-3,n-1}}{n^3} & \frac{c_{n-2,n-1}}{n^3} & \frac{5}{6n^3} & \frac{c_{n-1,n}}{8n^3} \\ \frac{c_{1,n}}{8n^3} & \frac{c_{2,n}}{8n^3} & \cdots & \frac{c_{n-3,n}}{8n^3} & \frac{c_{n-2,n}}{8n^3} & \frac{c_{n-1,n}}{8n^3} & \frac{1}{24n^3} \end{pmatrix} \tag{10}$$

Matrix Constructions $(\mathbf{M} \cdot \ddot{\phi} + \mathbf{K} \cdot (\phi + d\,\dot{\phi}) + \mathbf{N} \cdot \dot{\phi}^2 + \mathbf{K}_B \cdot \phi = \mathbf{F})$

2 **Linear stiffness matrix K**: This is an $n \times n$ symmetric matrix:

$$\mathbf{K} = n \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix} \tag{11}$$

Matrix Constructions ($\mathbf{M}\cdot\ddot{\phi}+\mathbf{K}\cdot(\phi+d\,\dot{\phi})+\mathbf{N}\cdot\dot{\phi}^2+\mathbf{K}_B\cdot\phi=\mathbf{F}$)

3. **Nonlinear matrix N**: This is an $n\times n$ skew-symmetric matrix representing a portion of the system's geometric nonlinearity:

$$\mathbf{N}=\begin{pmatrix} 0 & \frac{(n-1)s_{1,2}}{n^3} & \cdots & \frac{3s_{1,n-3}}{n^3} & \frac{2s_{1,n-2}}{n^3} & \frac{s_{1,n-1}}{n^3} & \frac{s_{1,n}}{8n^3} \\ -\frac{(n-1)s_{1,2}}{n^3} & 0 & \cdots & \frac{3s_{2,n-3}}{n^3} & \frac{2s_{2,n-2}}{n^3} & \frac{s_{2,n-1}}{n^3} & \frac{s_{2,n}}{8n^3} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ -\frac{3s_{1,n-3}}{n^3} & -\frac{3s_{2,n-3}}{n^3} & \cdots & 0 & \frac{2s_{n-3,n-2}}{n^3} & \frac{s_{n-3,n-1}}{n^3} & \frac{s_{n-3,n}}{8n^3} \\ -\frac{2s_{1,n-2}}{n^3} & -\frac{2s_{2,n-2}}{n^3} & \cdots & -\frac{2s_{n-3,n-2}}{n^3} & 0 & \frac{s_{n-2,n-1}}{n^3} & \frac{s_{n-2,n}}{8n^3} \\ -\frac{s_{1,n-1}}{n^3} & -\frac{s_{2,n-1}}{n^3} & \cdots & -\frac{s_{n-3,n-1}}{n^3} & -\frac{s_{n-2,n-1}}{n^3} & 0 & \frac{s_{n-1,n}}{8n^3} \\ -\frac{s_{1,n}}{8n^3} & -\frac{s_{2,n}}{8n^3} & \cdots & -\frac{s_{n-3,n}}{8n^3} & -\frac{s_{n-2,n}}{8n^3} & -\frac{s_{n-1,n}}{8n^3} & 0 \end{pmatrix} \quad (12)$$

Matrix Constructions ($\mathbf{M} \cdot \ddot{\phi} + \mathbf{K} \cdot (\phi + d\,\dot{\phi}) + \mathbf{N} \cdot \dot{\phi}^2 + \mathbf{K}_B \cdot \phi = \mathbf{F}$)

4. **Stiffness matrix at the boundary $\mathbf{K}_B$**: This matrix is obtained by summing the products of the total horizontal, vertical, and angular stiffness matrices with their respective stiffness coefficients (K_x^{Total}, K_y, K_ϕ):

$$\mathbf{K}_B = k_x^{Total}\mathbf{K}_x + k_y\mathbf{K}_y + k_\phi\mathbf{K}_\phi \tag{13}$$

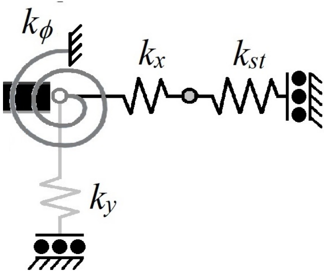
The right boundary is physically modeled by three springs: a horizontal spring (k_x), a vertical spring (k_y), and an angular spring (k_ϕ), therefore:

$$\mathbf{K}_y = ss_n \begin{pmatrix} c_1 & 0 & \cdots & 0 & 0 \\ 0 & c_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & c_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{2}c_n \end{pmatrix} \quad \mathbf{K}_\phi = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \tag{14}$$

Matrix Constructions ($\mathbf{M} \cdot \ddot{\phi} + \mathbf{K} \cdot (\phi + d\,\dot{\phi}) + \mathbf{N} \cdot \dot{\phi}^2 + \mathbf{K}_B \cdot \phi = \mathbf{F}$)

The total horizontal Stiffness Matrix $\mathbf{K}_x$ incorporates both the stretching effects and the stiffness of the horizontal boundary conditions:

$$\mathbf{K}_x = \left(\frac{2n-1}{2n^2} - sc_n\right) \begin{pmatrix} s_1 & 0 & \cdots & 0 & 0 \\ 0 & s_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & s_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{2}s_n \end{pmatrix} \tag{15}$$



Matrix Constructions ($\mathbf{M} \cdot \ddot{\phi} + \mathbf{K} \cdot (\phi + d\,\dot{\phi}) + \mathbf{N} \cdot \dot{\phi}^2 + \mathbf{K}_B \cdot \phi = \mathbf{F}$)

5. **External force vector F**: This is an $n \times 1$ vector representing the external force due to base excitation:

$$\mathbf{F} = F\cos(\Omega t)\begin{pmatrix} \frac{(n-1)c_1}{n^2} \\ \frac{(n-2)c_2}{n^2} \\ \vdots \\ \frac{c_{n-1}}{n^2} \\ \frac{c_n}{8n^2} \end{pmatrix} \tag{16}$$

6. **Internal damping matrix C**: The internal damping matrix is related to the damping coefficient d and the stiffness matrix, as can be realized from Eq. (8):

$$\mathbf{D} = d\,\mathbf{K}. \tag{17}$$

Building Matrices

The elements of these matrices are reformulated according to the row index $(1 \leq i \leq n)$, column index $(1 \leq j \leq n)$, and the size of the matrix (n) for ease of coding implementation, as follows:

- For $i \leq j$ & $i = j \ldots n-1$:

$$M_{i,j} = \frac{(n-j)c_{i,j}}{n^3}, \quad K_{i,i+1} = -n, \quad K_{i,i+j+1} = 0,$$
$$N_{i,j} = \frac{(n-j)s_{i,j}}{n^3}, \quad {K_x^{Total}}_{i,j} = 0, \quad {K_y}_{i,j} = 0,$$
$$K_{\phi_{i,j}} = 0. \tag{18}$$

- For $j \leq i$ & $i = j \ldots n$:

$$M_{j,i} = M_{i,j}, \quad K_{j,i} = K_{i,j}, \quad N_{j,i} = -N_{i,j},$$
$$K_{j,i} = K_{i,j}, \quad {K_x^{Total}}_{j,i} = {K_x^{Total}}_{i,j}, \quad {K_y}_{i,j} = {K_y}_{j,i},$$
$$K_{\phi_{j,i}} = K_{\phi_{i,j}}. \tag{19}$$

Building Matrices

- For $i = j$ & $i = 1 \dots n-1$:

$$M_{i,i} = \frac{6n - 6i - 1}{6n^3}, \quad K_{i,i} = 2n, \quad N_{i,i} = 0,$$
$$K_x^{Total}{}_{i,i} = s_i \left(\frac{2n-1}{2n^2} - sc_n \right), \quad K_{y\,i,i} = c_i ss_n,$$
$$K_{\phi_{i,i}} = 0, \quad F_{i,1} = \frac{(n-i)c_i}{n^2}. \tag{20}$$

- For $i = j = n$:

$$M_{n,n} = \frac{1}{24n^3}, \quad K_{n,n} = n, \quad N_{n,n} = 0,$$
$$K_x^{Total}{}_{n,n} = \frac{1}{2} s_n \left(\frac{2n-1}{2n^2} - sc_n \right), \quad K_{y\,n,n} = \frac{1}{2} c_n ss_n,$$
$$K_{\phi_{n,n}} = K_\phi, \quad F_{n,1} = \frac{c_n}{8n^2}. \tag{21}$$

Outline

Formulation of Euler-Bernoulli Problem

- Focus: C-C (Clamped-Clamped) and C-F (Clamped-Free) beams.
- Define variables: Position s, time τ, horizontal displacement U, vertical displacement W.
- Introduced non-dimensional variables:

$$u = \frac{U}{l}, \quad w = \frac{W}{l}, \quad x = \frac{s}{l}, \quad t = \frac{\tau}{T}, \quad T = \sqrt{\frac{\rho t_b w_b l^4}{EI}}$$

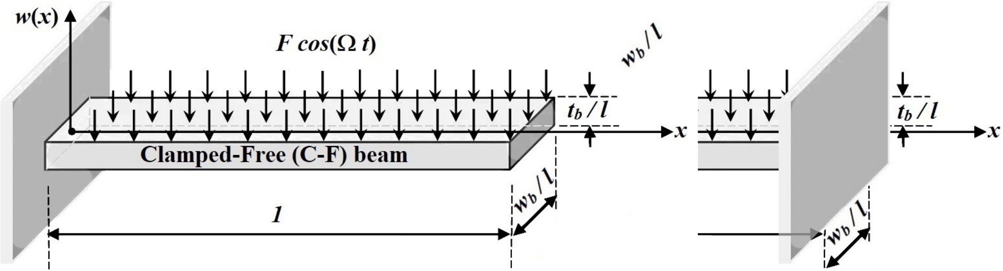


Figure 3: Physical models of the C-F and C-C beam systems

Beam Boundary Conditions and PDEs

Shortened-beam Boundaries (C-F): One edge free for sliding, inextensional.

$$e = 0 \quad \Rightarrow \quad (1+u')^2 + (w')^2 = 1, \tag{22}$$

$$PDE_{Sh} = \ddot{w} + w^{(iv)} + d\dot{w}^{(iv)} + \left(w'w''^2 + w'''w'^2\right)' - F\cos(\Omega t) = 0 \tag{23}$$

Stretched-beam Boundaries (C-C): Both ends fixed; longitudinal strain.

$$e \neq 0 \quad \Rightarrow \quad e = u' + \frac{1}{2}(w')^2 \tag{24}$$

$$PDE_{St} = \ddot{w} + w^{iv} + d\dot{w}^{iv} + [w'w''w''']' - \frac{1}{2}w'\left(\int_0^x w'^2\,dx\right)'' = 0 \tag{25}$$

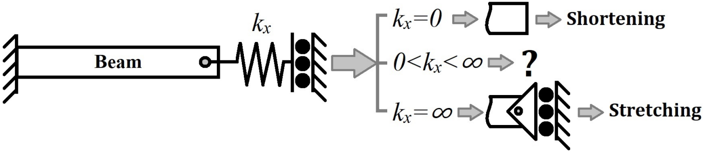


Figure 4: Configuration example of Partly-Shortened or Partly-stretched beam boundaries

Solving PDEs via Galerkin method

$$w(x,t) = \sum_{i=1}^{N_m} a_i(t)\psi_i(x), \tag{26}$$

$$\psi_i(x) = A_1\cosh(\sqrt{\omega_i}x) + A_2\sinh(\sqrt{\omega_i}x) + A_3\sin(\sqrt{\omega_i}x) + A_4\cos(\sqrt{\omega_i}x). \tag{27}$$

$w(x,t)$ becomes after normalizing the shape:

$$w(x,t) = \sum_{i=1}^{N_m} a_i^{Norm}(t)\psi_i^{Norm}(x), \tag{28}$$

Substituting into the PDEs:

1. For shortened-beam:

$$ODE_j = \int_0^1 \psi_j^{\text{Norm}}(x)\, PDE_{\text{Sh}}\, dx. \tag{29}$$

2. For stretched-beam:

$$ODE_j = \int_0^1 \psi_j^{\text{Norm}}(x)\, PDE_{\text{St}}\, dx. \tag{30}$$

Outline

System Properties

For validation, We selected a sample system similar to a widely available steel ruler beam, ensuring accessibility for future experimental investigations.

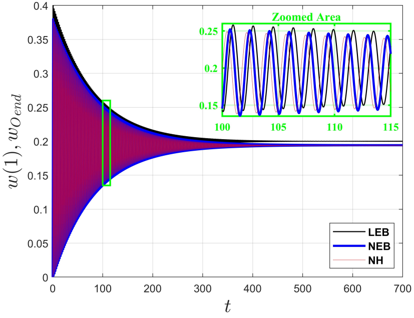
Table 1: Geometrical and mechanical properties of the considered system

l (cm)	w (cm)	t_b (cm)	E (GPa)	ρ_b (kg/m^3)	d
50	2.6	0.8	200	7850	0.002

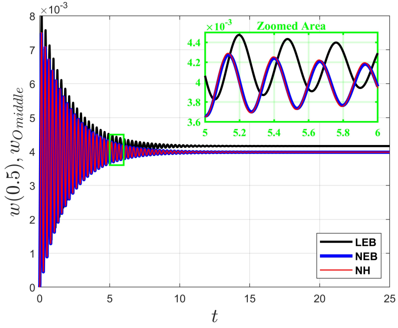
Table 2: Abbreviations

Abbreviation	**Description**
LEB	Linear Euler-Bernoulli model discretized via Galerkin method
NEB	Nonlinear Euler-Bernoulli model discretized via Galerkin method
NH	Nonlinear Hencky beam Model method

Validation Results: Static Analysis of Time History (n=25)



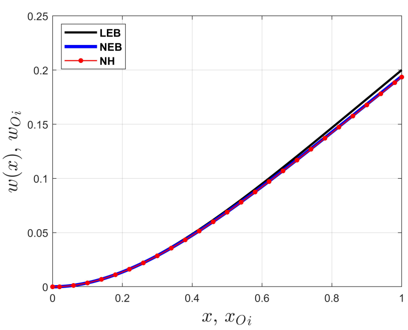
(a) C-F beam (midpoint)



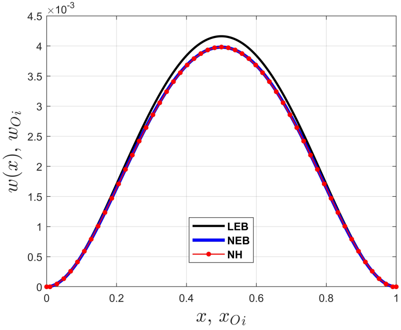
(b) C-C beam (midpoint)

Figure 5: Comparison of NH with 26 elements ($n = 25$), LEB, and NEB with 3 modes ($N_m = 3$): Static time history analysis ($\Omega = 0$) subjected to a force of $F = 1.6$.

Validation Results: Static Analysis of deflection (n=25)



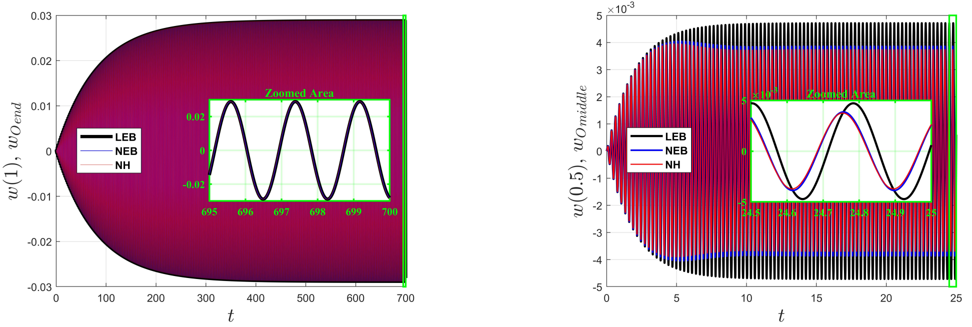
(a) C-F beam at $t = 700$.



(b) C-C beam at $t = 25$.

Figure 6: Comparison of NH with 26 elements ($n = 25$), LEB, and NEB with 3 modes ($N_m = 3$): Static deflection ($\Omega = 0$) after stabilization subjected to a force of $F = 1.6$.

Validation Results: Dynamic Analysis of Time History (n=25)



(a) C-F Beam (Endpoint) subjected to a force of $F = 0.0016$ from $t = 0$ to $t = 25$.

(b) C-C Beam (Midpoint) subjected to a force of $F = 0.08$ from $t = 0$ to $t = 700$.

Figure 7: Comparison of NH with 26 elements ($n = 25$), LEB, and NEB with 3 modes ($N_m = 3$): Time History Analysis Near the First Resonance Frequency ($\Omega = \omega_1$).

Validation Results: Dynamic Analysis of Time History (n=52)

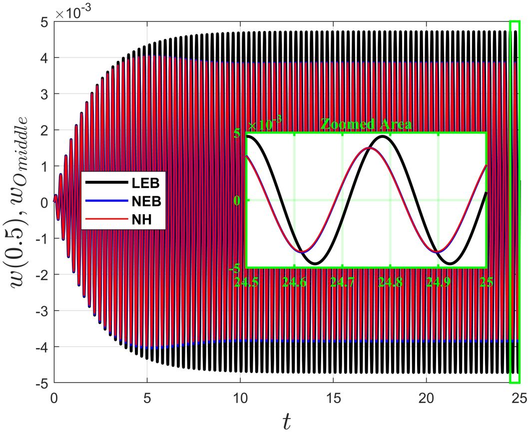
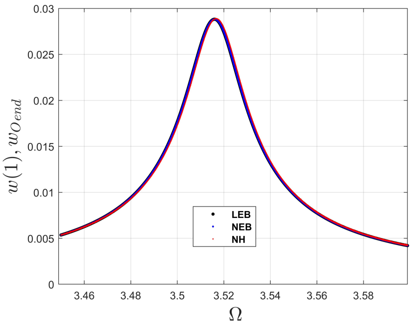
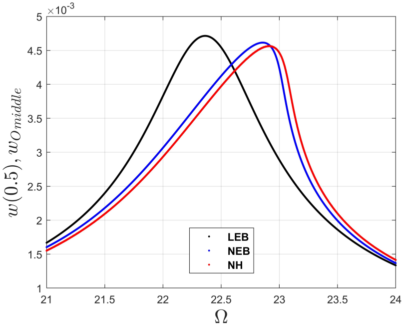


Figure 8: Comparison of NH with 52 elements (n=51), LEB, and NEB with 3 modes ($N_m = 3$): Time History Analysis near the first resonance frequency $\Omega = \omega_1$ for a C-C beam (midpoint) subjected to a force of $F = 0.08$ from $t = 0$ to $t = 700$.

Validation Results: Frequency Response Analysis (n=25)



(a) C-F beam (endpoint) subjected to a force of $F = 0.0016$.



(b) C-C beam (midpoint) subjected to a force of $F = 0.08$.

Figure 9: Comparison of NH with 26 elements ($n = 25$), LEB, and NEB with 3 modes ($N_m = 3$): Frequency Response Analysis near the first resonance frequency ($\Omega \approx \omega_1$).

Validation Results: Frequency Response Analysis (n=51)

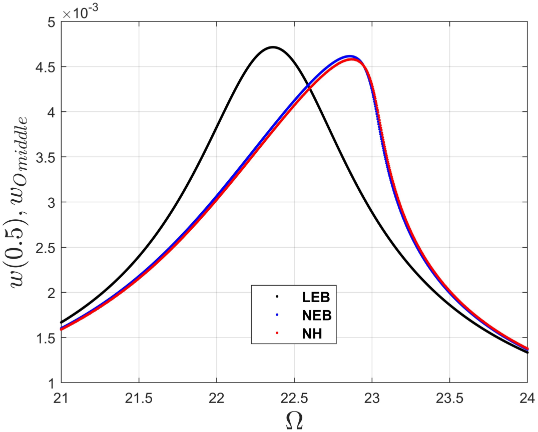


Figure 10: Comparison of NH with 52 elements ($n = 51$), LEB, and NEB with 3 modes ($N_m = 3$): Frequency Response Analysis near the first resonance frequency ($\Omega \approx \omega_1$) for C-C beam (midpoint) subjected to a force of $F = 0.08$.

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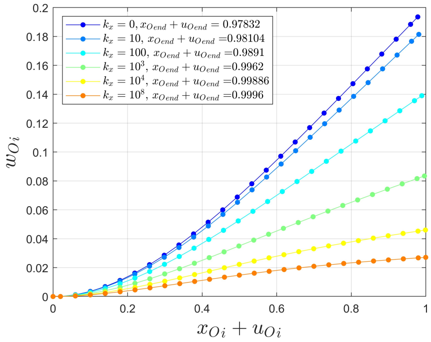
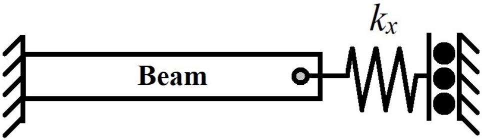
Applications: Nonlinear analysis for large deflections and certain supports

This section explores a configuration where the horizontal spring stiffness k_x is finite, making the system neither fully free nor clamped.



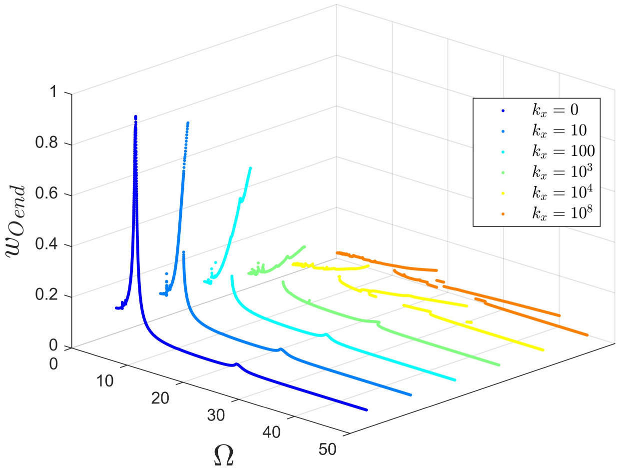
Static deflection after stabilization ($\Omega = 0$, $F = 1.6$, $t = 700$) for varying k_x. Increasing k_x reduces static deflection.



Schematic of a clamped-free beam with a spring (k_x) at the free end.

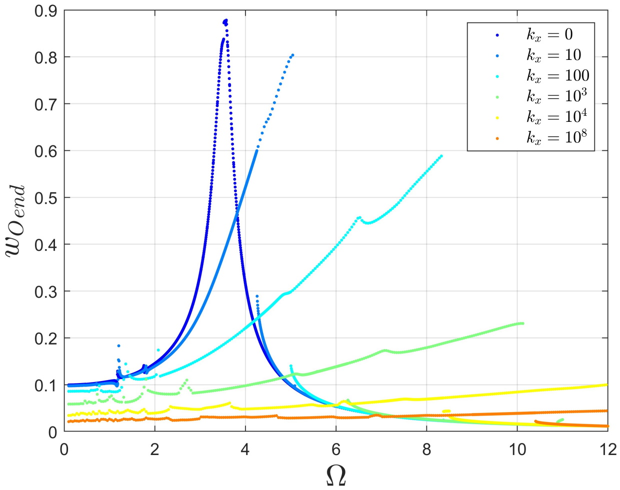
Figure 11: Effect of spring stiffness k_x on beam behavior.

Dynamic response: Influence of k_x



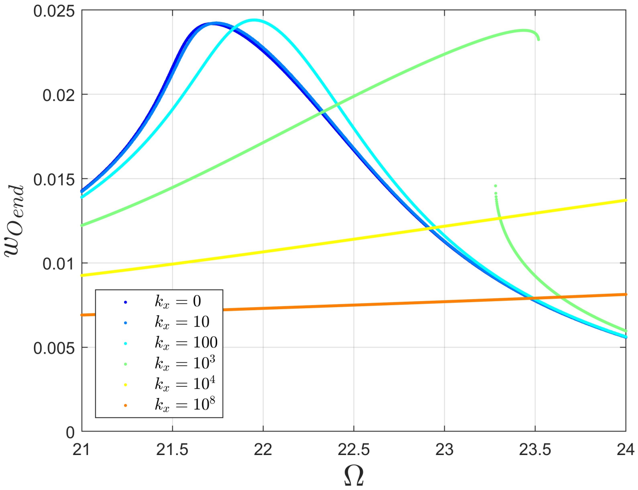
Frequency response for $0 \leq \Omega \leq 45$ and varying k_x. First and second resonances both shift rightward and reduce in magnitude as k_x increases.

Zoom: First resonance region



Frequency response near first resonance ($0 \le \Omega \le 10$). Hardening behavior increases with k_x.

Zoom: Second resonance region



Frequency response near second resonance ($21 \leq \Omega \leq 24$). Behavior shifts from softening ($k_x < 100$) to hardening ($k_x > 100$).

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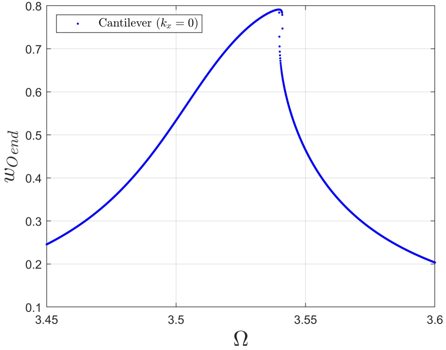
Cantilever response: First resonance

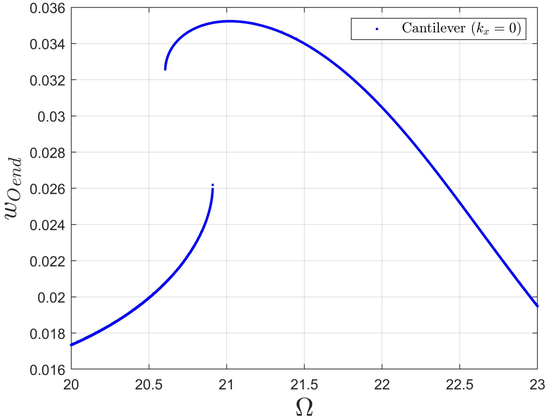
Nayfeh and Pai confirm these findings in their study of the nonlinear behavior of cantilever beams. They discovered that this nonlinear behavior arises from a combination of hardening and softening effects. The overall nonlinear response depends on the balance between these two effects. Specifically, the first resonance mode demonstrates hardening,



Frequency response near $\omega_1 = 3.52$ with $F = 0.08$ for cantilever ($k_x = 0$). Hardening behavior is evident.

Cantilever response: Second resonance

, whereas higher modes tend to exhibit softening [3].



Frequency response near $\omega_2 = 22.03$ with $F = 1.6$ for cantilever ($k_x = 0$). Clear softening behavior observed.

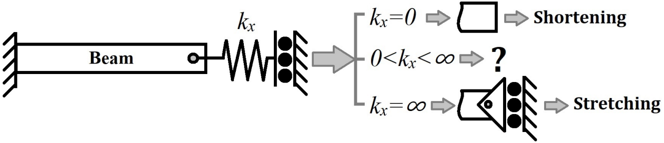
Outline

Conclusion

- Validation using the Galerkin method confirmed that Hencky's models are effective for both cantilever and clamped-clamped beams.
- Hencky's models overcome the limitations of the nonlinear Euler-Bernoulli:
  1. The Euler-Bernoulli model only applies to small deflections, while Hencky's models handle both small and large deflections.
  2. The Euler-Bernoulli model struggles with partially shortened or stretched boundaries, but Hencky's models do not have these boundary restrictions.

Future Research:

- Despite the complexity, Hencky's models can be applied to nonlinear dynamic studies of beams with partially shortened or stretched boundaries.



- Explore nonlinear behaviors in beams with varying boundary conditions, such as different slope angles.

Outline

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Thank You for Your Attention!

Questions or Comments?