Challenges and Solutions in Analytical Approaches for Nonlinear Systems

MSc: Muhammad Umer
Supervisor: Prof. Paweł Olejnik

Department of Automation, Biomechanics and Mechatronics,
Faculty of Mechanical Engineering, Lodz University of Technology, Lodz.

03 June, 2025

Outline

Regular Perturbation Method

Consider a Duffing oscillator, which we solve using the Regular Perturbation Method (RPM), also known as the Straightforward Method. This problem shows an interesting challenge because it is singularly perturbed; in other words, its natural frequency is equal to one.

Example

We solve a nonlinear second-order ODE, where the nonlinearity arises from the cubic term y^3, and ϵ is a small parameter:

$$y'' + y + \epsilon y^3 = 0 \tag{1}$$

Example

Solution:
Apply 2nd law of Newton and rearrange the Equation (1), we get

$$F(y) = -y - \epsilon y^3 \tag{2}$$

where $F(y)$ is a restoring force for a weakly non-linear spring. Here, we have three cases of ϵ,

- if $\epsilon > 0$ then the spring is hard.
- if $\epsilon = 0$ then the spring is linear.
- if $\epsilon < 0$ then the spring is soft.

Let, our expected solution $y(t, \epsilon)$ to be periodic in "t" (since no damping and external force present). Now, we try to approximate "y" for small ϵ i.e

$$y(t, \epsilon) = y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t) + \ldots \tag{3}$$

Example

Now, we plugged Equation (3) into (1) also neglect ϵ^2 and higher powers,

$$(y_0''(t)+\epsilon y_1''(t)+\ldots)+(y_0(t)+\epsilon y_1(t)+\ldots)+\epsilon(y_0(t)+\epsilon y_1(t)+\ldots)^3 = 0 \tag{4}$$

Compare the powers of order 1 and order ϵ we get,

$$O(1) : y_0'' + y_0 = 0 \tag{5}$$

$$O(\epsilon) : y_1'' + y_1 + y_0^3 = 0 \tag{6}$$

Now, we solve both Equations (5) and (6) one by one. The initial conditions for Equation (5) are $y_0(0) = 1$ and $\dot{y}_0(0) = 0$. Applying these initial conditions, we obtain:

$$y_0(t) = \cos(t). \tag{7}$$

Example

Next, we consider Equation (6) and rearrange it. Substituting the value of $y_0(t)$ into the left side gives us:

$$\ddot{y}_1 + y_1 = -\cos^3(t). \tag{8}$$

We can express $\cos^3(t)$ using trigonometric identities, which allows us to rewrite Equation (8) in the form:

$$\ddot{y}_1 + y_1 = -\frac{1}{4}\cos(3t) - \frac{3}{4}\cos(t). \tag{9}$$

Now, we have two cases first solve the R.H.S of Equation (9) for $-\frac{3}{4}\cos(t)$ and then for $-\frac{1}{4}\cos(3t)$ respectively. For both cases Equation (9) is nonhomogeneous D.E. So, we solved it by method of undetermined coefficient. In this method we shall find the solution of given D.E as

$$f(D)y_1 = g(x) \tag{10}$$

Example

Its general solution will be

$$y_1(t) = y_c + y_p \tag{11}$$

where y_c is determined by the homogeneous part of the equation. For y_p we assume a specific form for the solution, which will include unknown coefficients that we can solve for using various techniques based on the form of $g(x)$.
First, form Equation (9), we find y_c by setting $\ddot{y}_1 + y_1$=0:

$$y_c = A\cos(t) + B\sin(t) \tag{12}$$

where A and B are constants.
Next, we assume a particular solution in the following representation:

$$y_p = t^k\left(c_1\cos(t) + c_2\sin(t)\right) . \tag{13}$$

Example

Therefore, y_p for the first case of Equation (9) becomes:

$$y_p = -\frac{3}{8} t \sin(t)\,. \tag{14}$$

For the second case of Equation (9), we obtain:

$$y_p = \frac{1}{32} \cos(3t)\,. \tag{15}$$

By combining y_c and y_p in Equation (11):

$$y_1(t) = A\cos(t) + B\sin(t) + \frac{1}{32}\cos(3t) - \frac{3}{8} t \sin(t). \tag{16}$$

Example

Apply the initial conditions and solve Equation (16) after simplifying the results, one finds $A = -\frac{1}{32}$ and $B = 0$. Thus,

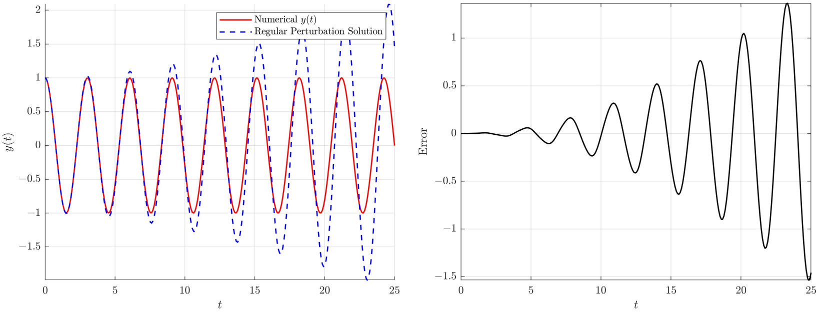
$$y_1(t) = \frac{1}{32}\left(\cos(3t) - \cos(t)\right) - \frac{3}{8} t \sin(t). \tag{17}$$

Finally, substituting $y_0(t)$ and $y_1(t)$ into Equation (3), we obtain:

$$\begin{aligned} y(t,\epsilon) &= \cos(t) \\ &+ \epsilon\left[\frac{1}{32}(\cos(3t) - \cos(t)) - \frac{3}{8} t \sin(t)\right] + O(\epsilon^2). \end{aligned} \tag{18}$$

The approximate analytical solution, as given by Equation (18), and the numerical solution modeling the Duffing oscillator expressed by Equation (1) are presented in Figure 1.

Figures



(a) Comparison of solutions **(b)** The error in time

Figure: Approximate analytical and numerical solution of Duffing oscillator for $\epsilon = 0.1$.

Conclusion of RP Method

Concluding, the term $-\frac{3}{4}\cos(t)$ causes resonance because $\cos(t)$ satisfies the homogeneous solution, leading to a secular term of $-\frac{3}{8}t\sin(t)$, as shown in Figure 1a. The response grows indefinitely due to resonance, becoming unbounded as $t \to \infty$. Consequently, the error between the analytical and numerical solutions increases rapidly, as illustrated in Figure 1b. A fundamental weakness in regular perturbation theory is the assumption that the frequency is given by the unperturbed frequency $\omega = 1$, which leads to inaccuracies in the analysis.

Poincaré-Lindstedt Method

Now we solve the same Duffing oscillator with the help of Poincaré-Lindstedt Method.

Example

We solve a nonlinear second-order ODE, where the nonlinearity arises from the cubic term y^3, and ϵ is a small parameter:

$$\ddot{y} + y + \epsilon y^3 = 0. \tag{19}$$

Example

Solution:
The above Equation (19) can be interpreted as a perturbed harmonic oscillator equation, the solution of which can be found as follows. Let

$$\tau = \omega t, \tag{20}$$

where t is the stretched time and $\omega = \omega(\epsilon)$ is a power series such that $\omega = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \dots$. We will determine ω by insisting that it is the true frequency, and hence the solution is 2π-periodic in τ. Converting the problem to one, we can solve it using the RPM, we have:

$$\dot{y} = \frac{dy}{dt} = \frac{dY}{d\tau}\frac{d\tau}{dt} = \omega\dot{Y}. \tag{21}$$

Similarly, we have:

$$\ddot{y} = \omega^2\ddot{Y}. \tag{22}$$

. Now, the differential equation takes the form:

$$\omega^2\ddot{Y} + Y + \epsilon Y^3 = 0, \tag{23}$$

Example

Putting the approximation and ω into Equation (23) and comparing the powers of order 1 and ϵ. After that we solve order 1 nad ϵ, we obtain:

$$Y_0(\tau) = \cos(\tau). \tag{24}$$

$$\ddot{Y}_1 + Y_1 = -\cos^3(\tau) + 2\omega_1 \cos(\tau). \tag{25}$$

Using trigonometric identities to express $\cos^3(\tau)$, we substitute it into Equation (25) and rearrange the terms, yielding:

$$\ddot{Y}_1 + Y_1 = -\frac{1}{4}\cos(3\tau) - \left(\frac{3}{4} - 2\omega_1\right)\cos(\tau). \tag{26}$$

The term $\cos(\tau)$ will cause resonance (i.e., produce a secular term) unless we eliminate it by choosing $\omega_1 = \frac{3}{8}$. This choice of ω_1 allows us to avoid the secular term. Substituting the values of ω_0 and ω_1, we find:

$$\omega = 1 + \frac{3}{8}\epsilon + O(\epsilon^2). \tag{27}$$

This choice ensures that the solution is 2π-periodic in τ.

Example

Next, we solve Equation (26), which is a nonhomogeneous differential equation. After solving it, we get:

$$Y_1 = \frac{1}{32}\left(\cos(3\tau) - \cos(\tau)\right). \tag{28}$$

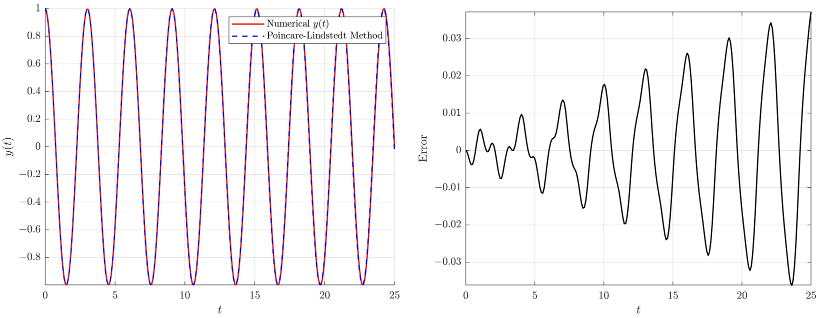
Substituting the values of Y_0 and Y_1, we obtain:

$$Y(\tau) = \cos(\tau) + \frac{1}{32}\left(\cos(3\tau) - \cos(\tau)\right) + O(\epsilon^2). \tag{29}$$

This equation indicates that there are no secular terms present, ensuring that the solution remains bounded for all values of τ. Now, substituting $\tau = \left(1 + \frac{3}{8}\epsilon\right) t$ into Equation (29), we find:

$$Y(t) \approx \cos\left[\left(1 + \frac{3}{8}\epsilon\right) t\right] + O(\epsilon). \tag{30}$$

Figures



(a) Comparison of solutions **(b)** Error

Figure: Approximate analytical and numerical solutions of the Duffing oscillator, as described by Equation (19), for $\epsilon = 0.1$.

Conclusion of PL Method

Concluding, as demonstrated in the example, the PLM is ineffective for analyzing stability or transient behavior. However, it serves as a valuable method for generating asymptotic approximations for periodic solutions, as illustrated in Figure 2a. Although the error shown in Figure 2b gradually increases, the values are very small compared to the observations in Figure 1b. The method is not suitable for obtaining solutions that evolve in an aperiodic manner over slow time scales.

Multiple Scale Method

Example

Consider a linear weekly damped oscillator and apply MMS for approximate analytical solutions:

$$\ddot{x} + 2\epsilon\dot{x} + x = 0. \quad (31)$$

Example

Solution:

Let

$$T_0 = t, \quad T_1 = \epsilon t, \tag{32}$$

where T_0 is a fast time and T_1 is a slow time.

Instead of establishing x as a function of t, we establish x as a function of both, T_0 and T_1, i.e.

$$x\,(t; \epsilon) = x\,(T_0, T_1; \epsilon)\,. \tag{33}$$

It should be noted that as the actual time t increases, the fast time T_0 increases at the same rate, while the slower time T_1 increases more gradually. Applying the chain rule, we get:

$$\frac{d}{dt}x(T_0, T_1) = \frac{\partial x}{\partial T_0}\frac{\partial T_0}{\partial t} + \frac{\partial x}{\partial T_1}\frac{\partial T_1}{\partial t}. \tag{34}$$

Example

As we know that from Equation (32), $\frac{\partial T_0}{\partial t} = 1$ and $\frac{\partial T_1}{\partial t} = \epsilon$. Putting these into Equation (34) and solving, we get:

$$\dot{x} = \frac{dx}{dt} = \frac{\partial x}{\partial T_0} + \epsilon \frac{\partial x}{\partial T_1}. \tag{35}$$

Similarly, for the second derivative, we obtain:

$$\ddot{x} = \frac{d^2x}{dt^2} = \frac{\partial^2 x}{\partial T_0^2} + 2\epsilon \frac{\partial^2 x}{\partial T_0 \partial T_1} + \epsilon^2 \frac{\partial^2 x}{\partial T_1^2}. \tag{36}$$

Now, substituting Equations (35) and (36) into Equation (31) and neglecting ϵ^2 and higher powers, we get:

$$\frac{\partial^2 x}{\partial T_0^2} + 2\epsilon \frac{\partial^2 x}{\partial T_0 \partial T_1} + 2\epsilon \frac{\partial x}{\partial T_0} + x = 0. \tag{37}$$

Considering the approximation:

$$x(t) = x\left(T_0, T_1\right) \sim \left(x_0\left(T_0, T_1\right) + \epsilon x_1\left(T_0, T_1\right) + \dots\right), \tag{38}$$

Example

we substitute Equation (38) into (37), while neglecting ϵ^2 and higher powers. After that:

$$\epsilon\left(\frac{\partial^2 x_1}{\partial T_0^2}+2\frac{\partial^2 x_0}{\partial T_0 \partial T_1}+2\frac{\partial x_0}{\partial T_0}+x_1\right)+\left(\frac{\partial^2 x_0}{\partial T_0^2}+x_0\right)=0. \quad (39)$$

We compare powers of order 1 and ϵ. For order $O(1)$, we get:

$$O(1): \quad \frac{\partial^2 x_0}{\partial T_0^2}+x_0=0. \quad (40)$$

$$O(\epsilon): \quad \frac{\partial^2 x_1}{\partial T_0^2}+2\frac{\partial^2 x_0}{\partial T_0 \partial T_1}+2\frac{\partial x_0}{\partial T_0}+x_1=0. \quad (41)$$

Solving Equation (40), we have:

$$x_0 = A(T_1)\cos(T_0)+B(T_1)\sin(T_0), \quad (42)$$

where A and B are constant with respect to T_0, but are functions of T_1.

Example

Rearranging Equation (41), we get:

$$\frac{\partial^2 x_1}{\partial T_0^2} + x_1 = -2\left(\frac{\partial^2 x_0}{\partial T_0 \partial T_1} + \frac{\partial x_0}{\partial T_0}\right). \tag{43}$$

After finding the derivatives of $\frac{\partial x_0}{\partial T_0}$ and $\frac{\partial^2 x_0}{\partial T_0 \partial T_1}$ and substituting them into Equation (43), simplifying and rearranging, we obtain the final form of the equation:

$$\frac{\partial^2 x_1}{\partial T_0^2} + x_1 = 2\left(\frac{\partial A}{\partial T_1} + A(T_1)\right)\sin(T_0) - 2\left(\frac{\partial B}{\partial T_1} + B(T_1)\right)\cos(T_0). \tag{44}$$

Here, $\left(\frac{\partial A}{\partial T_1} + A(T_1)\right)$ and $\left(\frac{\partial B}{\partial T_1} + B(T_1)\right)$ are secular terms that cause resonance. To eliminate these terms, we set them equal to zero and solve the resulting equations. After solving, we find:

$$A = ae^{-T_1}, \quad B = be^{-T_1}. \tag{45}$$

Example

Substituting the values of A and B into Equation (42), we get:

$$x_0 = ae^{-T_1}\cos(T_0) + be^{-T_1}\sin(T_0). \tag{46}$$

Now, we have the initial conditions:

- IC1: $x_0(0,0) = 1$,
- IC2: $\frac{\partial x_0(0,0)}{\partial T_0} = 0$.

Now, we solve Equation (46) using IC1 and IC2. We find $a = 1$ and $b = 0$. Putting these values into Equation (46) yields:

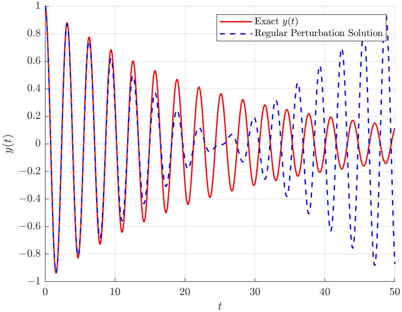
$$x_0 = e^{-T_1}\cos(T_0). \tag{47}$$

Next, we substitute x_0 from Equation (47) into Equation (38), and replace T_0 and T_1 with their original variables t and ϵt. Finally, we find:

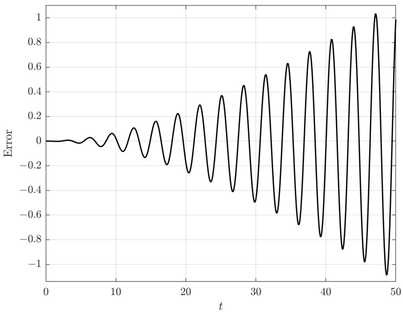
$$x = e^{-\epsilon t}\cos(t) + O(\epsilon), \tag{48}$$

being uniformly valid for $t = O(\frac{1}{\epsilon})$.

Figures



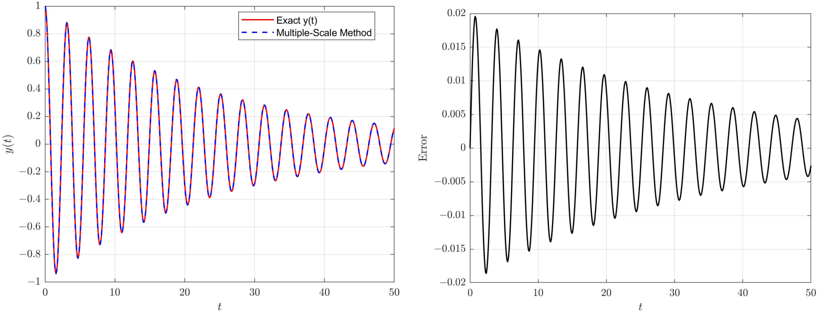
(a) Comparison of solutions



(b) The error in time

Figure: Approximate analytical RPM and exact solution of linear damped mass-spring system with no external forces for $\epsilon = 0.02$.

Figures



(a) Comparison of solutions (b) The error in time

Figure: Approximate analytical MMS and exact solution of linear damped mass-spring system with no external forces for $\epsilon = 0.02$.

Conclusion of MS Method

First, we compare the Regular Perturbation Method (RPM). The RPM solution (Figure 3a, dashed line) diverges from the numerical solution after approximately 23 seconds, tending toward infinity. Figure 3b highlights the steady asymptotic growth of the estimation error.

In Figure Figure 4a, applying MMS shows convergence with the numerical solution, while Figure 4b shows a decreasing error after each oscillation, confirming the superior efficiency of the MS method.

Table A5: Advantages and Disadvantages of Methods for Solving Nonlinear DEs

Method	**Advantages**	**Disadvantages**
Perturbation Method	Simplifies complex nonlinear equations, provides insight into weakly nonlinear systems, and is applicable for small deviations.	Resonance cause secular terms, only work well for single scale problem, unsuitable for strongly nonlinear systems.
Poincaré-Lindstedt Method	Eliminates secular terms, effective for periodic solutions.	Ineffective for stability analysis or transient behavior, not suitable for aperiodic solutions.
Method of Multiple Scales	Captures dynamics across multiple scales, avoid secular terms, versatile for nonlinear systems.	Challenging to identify relevant scale, computationally intensive, series convergence not guaranteed.

Table A3: Real World Applications of Methods for Solving Nonlinear DEs

Method	Applications
Perturbation Method	Duffing oscillators, nonlinear Schrödinger equation [19], Korteweg-de Vries equation [18], mechanical vibrations, quantum mechanics, and fluid dynamics.
Poincaré-Lindstedt Method	Non-linear vibration problem of multilayer plates consisting of NHOLs [164], linear damped mass-spring system with no external forces.
Method of Multiple Scales	Mechanical parametric oscillator with dry friction [62], 4-DOF variable-length pendulum [73], a piezoelectric transducer embedded in a nonlinear damped dynamical system [165].

References

- Nayfeh, A. H.: Perturbation Methods. Wiley-VCH, Weinheim (2008).
- Andrianov, I. V., & Awrejcewicz, J. (2024). Asymptotic methods for engineers. CRC Press.
- Holmes, M. H. (2012). Introduction to perturbation methods (Vol. 20). Springer Science & Business Media.
- Yakubu, G., & Olejnik, P. (2024). AN APPROXIMATE ANALYTICAL SOLUTION OF A 4-DOF VARIABLE-LENGTH PENDULUM MODEL. Journal of Theoretical & Applied Mechanics (14292955), 62(3).

Thanks!