# Filippov Methods for Non-smooth Systems

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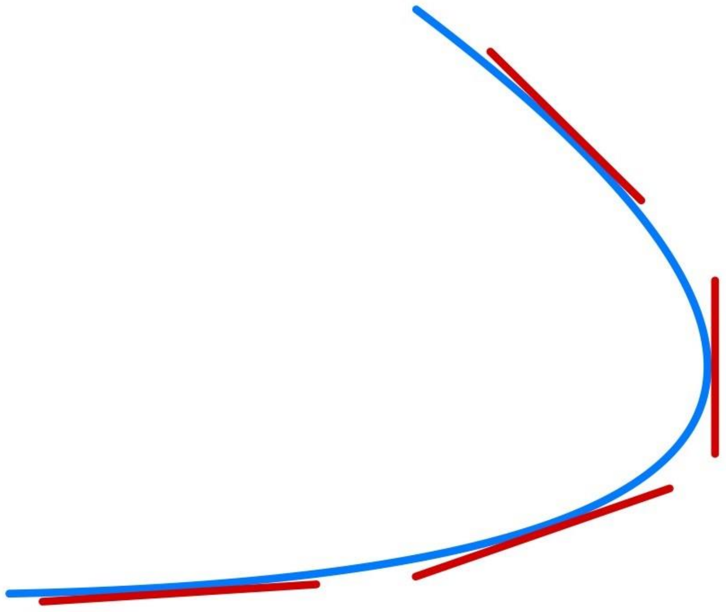
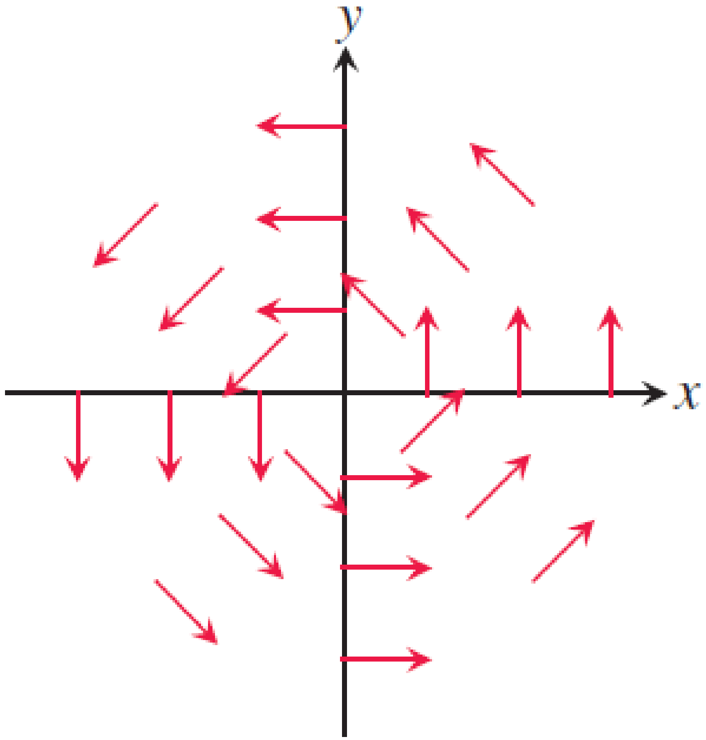
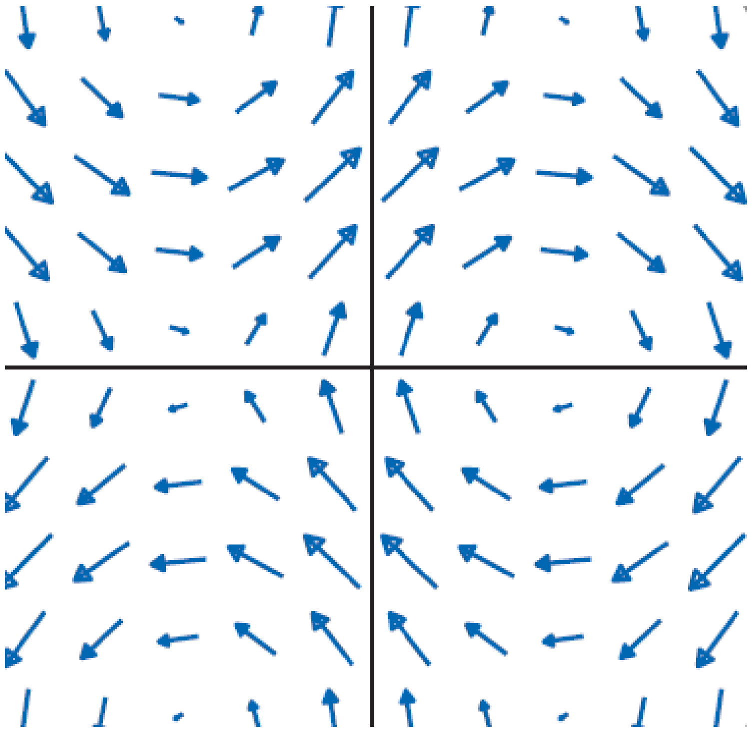
# Vector Fields and Orbits



Tangents to a curve

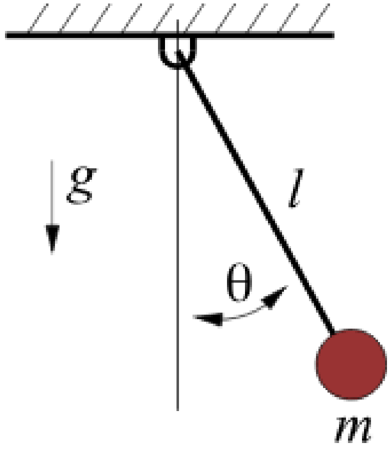



Velocity field



Velocity field

# Differential Equations as Inverse Problems



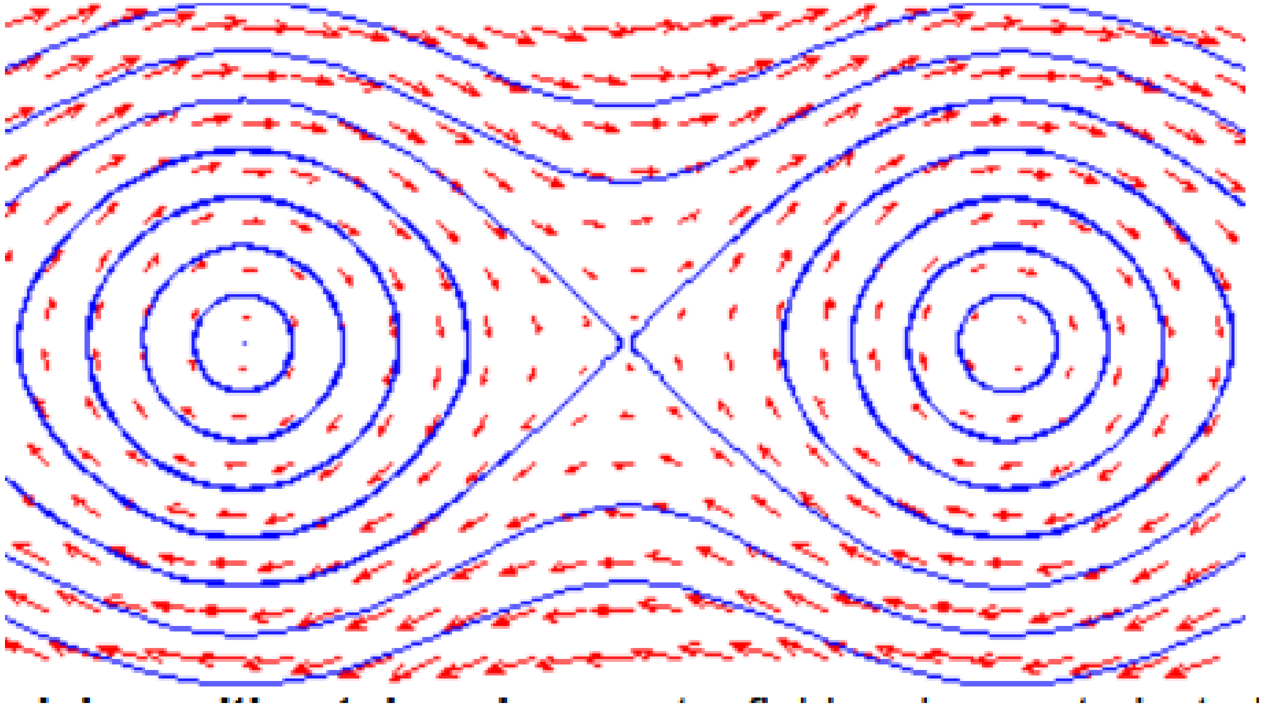

Governing equation: $\ddot{\theta} + \frac{g}{l}\sin\theta = 0$

State space form with $\theta = x$ and $\dot{\theta} = y$

$$\dot{x} = y$$

$$\dot{y} = -\frac{g}{l}\sin x$$

The velocity vector in state space: $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} y \\ \frac{-g}{l}\sin x \end{bmatrix}$



Vector field and some typical orbits of nonlinear pendulum

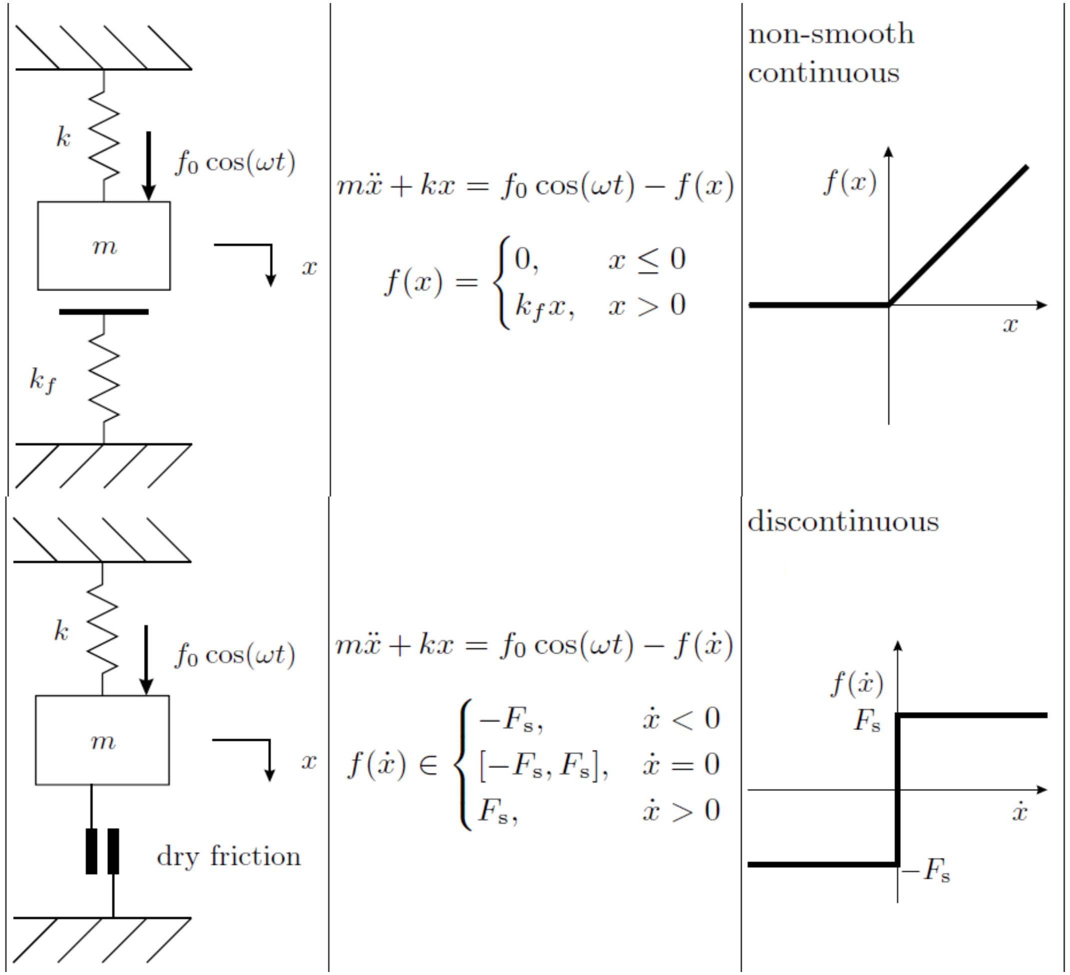
# Non-smooth and Discontinuous Vector Fields

Dynamical systems of the form:

$$\dot{X} = f(X, \dot{X}, t)$$

where, vector field $f$ is non-smooth and/or discontinuous.

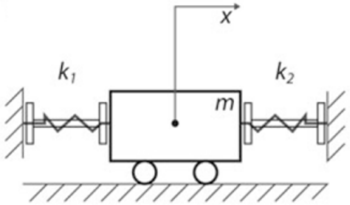
*R. I. Leine, H. Nijmeijer. Dynamics and Bifurcations of Non-Smooth Mechanical Systems, Springer Verlag, Berlin, 2004.*



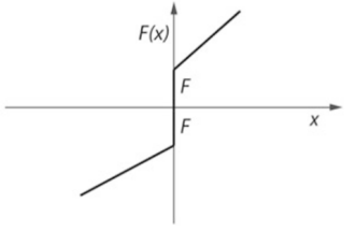
# Filippov Systems

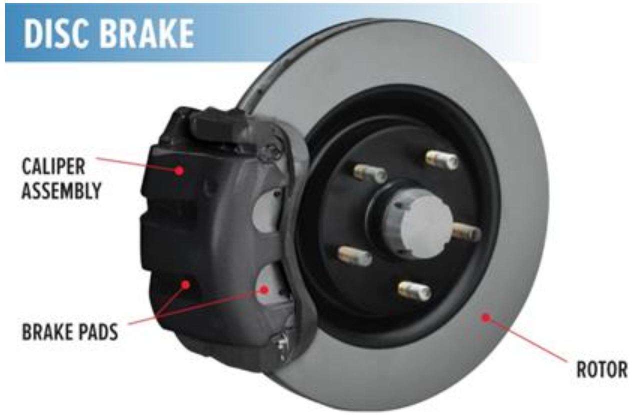
A Filippov system is characterized by

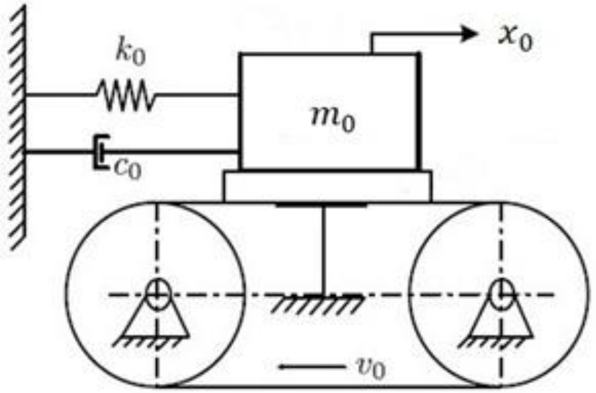
- Discontinuous vector field *f*
- Continuous, but possibly non-smooth orbit *X*

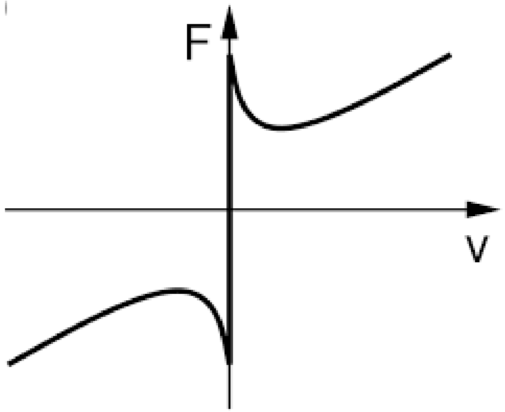



$$F(x) = \begin{cases} k_2 x + F_0 & \text{for } x > 0, \\ 0 & \text{for } x = 0, \\ k_1 x - F_0 & \text{for } x < 0. \end{cases}$$




*Jan Awrejcewicz. Ordinary Differential Equations and Mechanical Systems, Springer, 2014.*

# Surface of Discontinuity/Sliding Surface

The governing equation in non-dimensional form:

$$\frac{d^2x}{dt^2} + 2\xi\frac{dx}{dt} + x + \mu_s \operatorname{sgn}(\dot{x} - v_b) - k_1(\dot{x} - v_b) + k_3(\dot{x} - v_b)^3 = 0$$

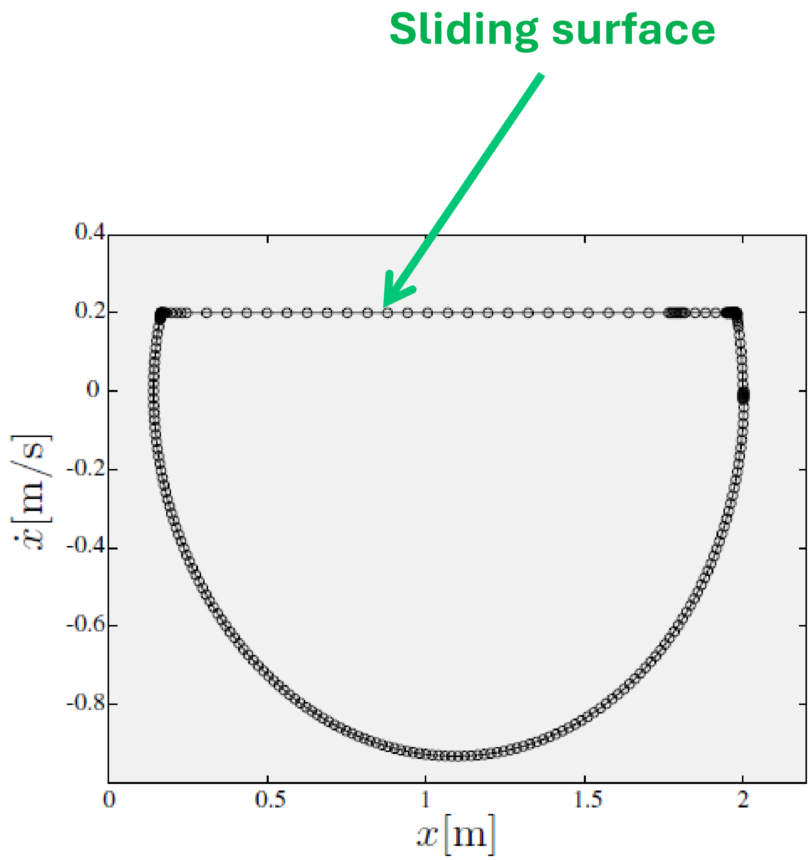
When $\dot{x} > v_b$

$$\frac{d^2x}{dt^2} + 2\xi\frac{dx}{dt} + x + \mu_s - k_1(\dot{x} - v_b) + k_3(\dot{x} - v_b)^3 = 0$$

When $\dot{x} < v_b$

$$\frac{d^2x}{dt^2} + 2\xi\frac{dx}{dt} + x - \mu_s - k_1(\dot{x} - v_b) + k_3(\dot{x} - v_b)^3 = 0$$

What is the vector field when $\dot{x} = v_b$?



# Partitioning of State space

System given by a finite set of ODEs

$$\dot{x} = F_i(x, \mu) \qquad \text{for} \qquad x \in S_i$$

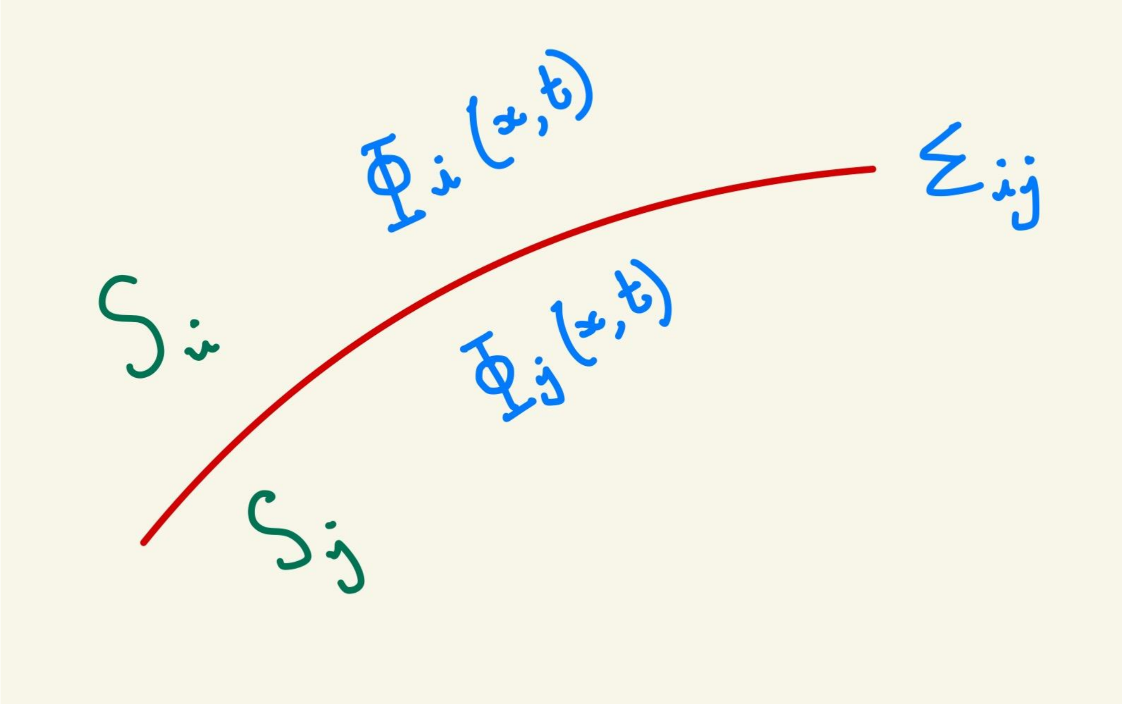
State space is partitioned into subsets $S_i$.

Each $F_i$ is smooth a defines a smooth flow $\Phi_i(x, t)$ in $S_i$.

$$\Sigma_{ij} = \overline{S_i} \cap \overline{S_j}$$

(Surface of discontinuity/Discontinuity Boundary/

Switching Manifold/Sliding Surface)



# Systems with One Sliding Surface

The single boundary $\Sigma$ given by the zero set of a smooth function $H$:

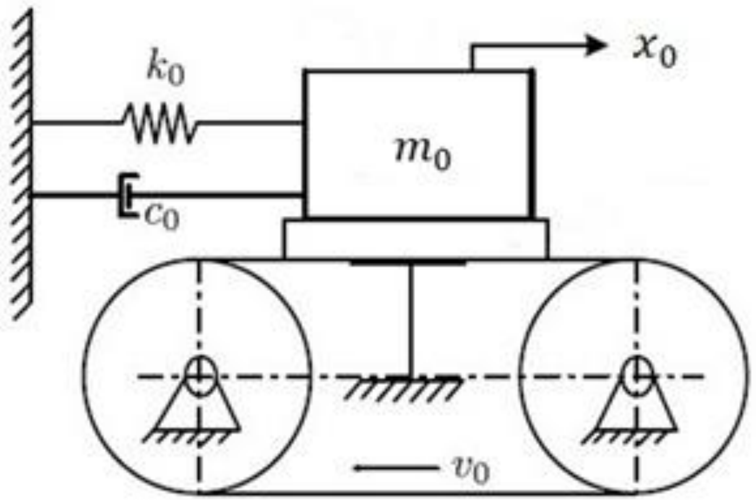
$$\Sigma = \{x \colon H(x) = 0\}$$

The partition of the state space becomes

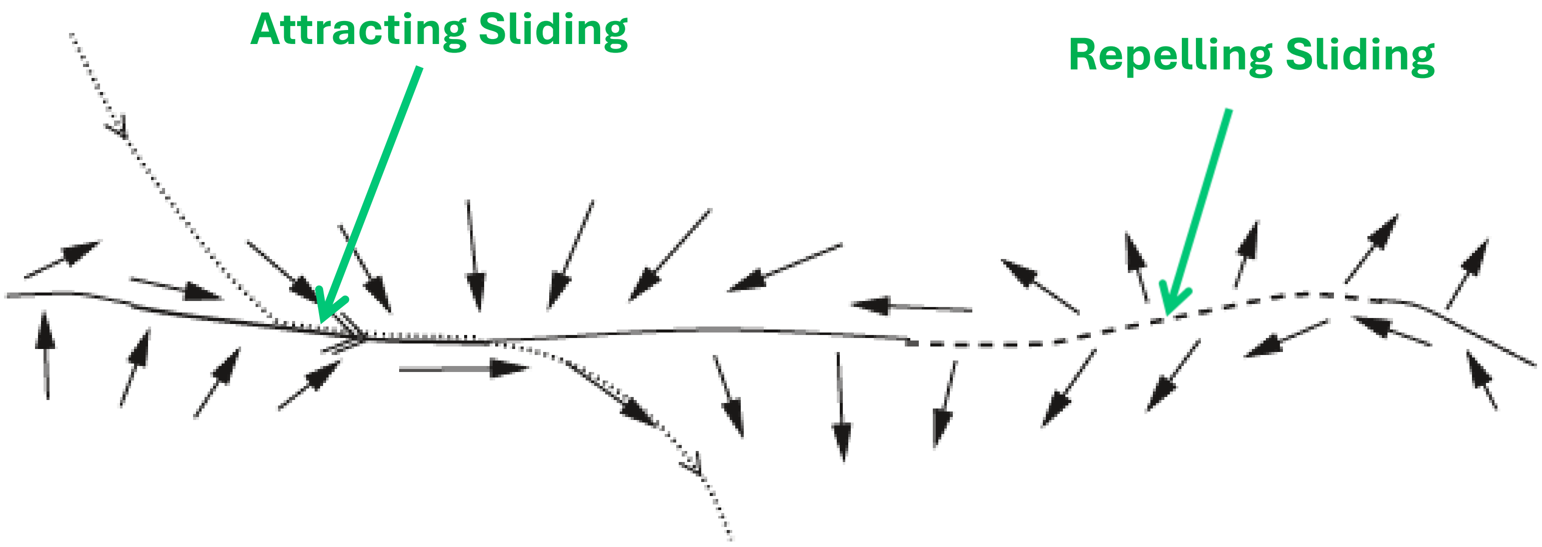
$$\dot{x} = \begin{cases} F_1(x), & H(x) > 0, \\ F_2(x), & H(x) < 0, \end{cases}$$

For mass-on-moving belt,

$$H(x) = \dot{x} - v_b$$



# Sliding Region on Discontinuity Surface




The orbit slides when the vector fields $F_1$ and $F_2$ on both sides of $\Sigma$ act in opposite directions:

$$(H_x F_1).(H_x F_2) < 0$$

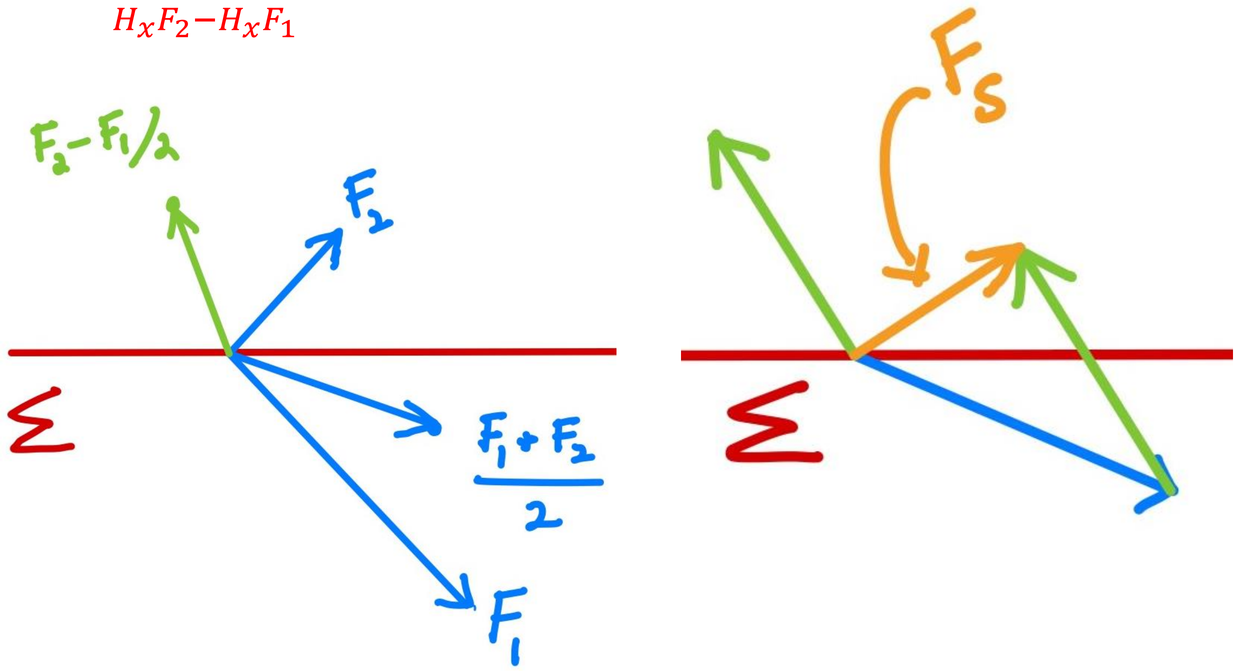
What is the vector field on the sliding region?

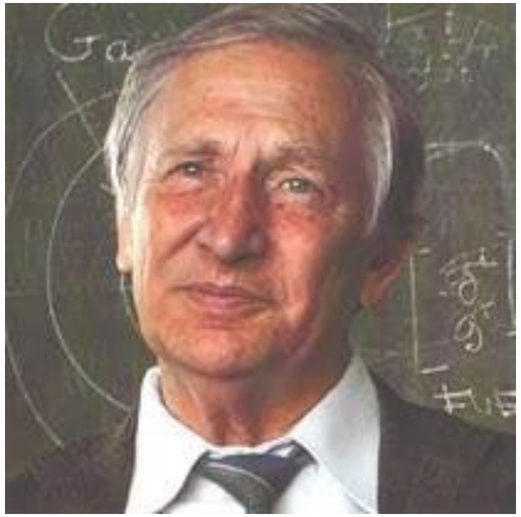
# Utkin's Equivalent Control Method

Utkin proposed the following law for the sliding vector field:

$$F_s = \frac{F_1+F_2}{2} + \beta\frac{F_2-F_1}{2}$$

where, $\beta = -\frac{H_x F_1 + H_x F_2}{H_x F_2 - H_x F_1}$



**Vadim Ivanovich Utkin**

Component of $F_s$ along the normal of $H$ is set to zero

$$\Longrightarrow H_x\left(\frac{F_1+F_2}{2} + \beta\frac{F_2-F_1}{2}\right) = 0$$

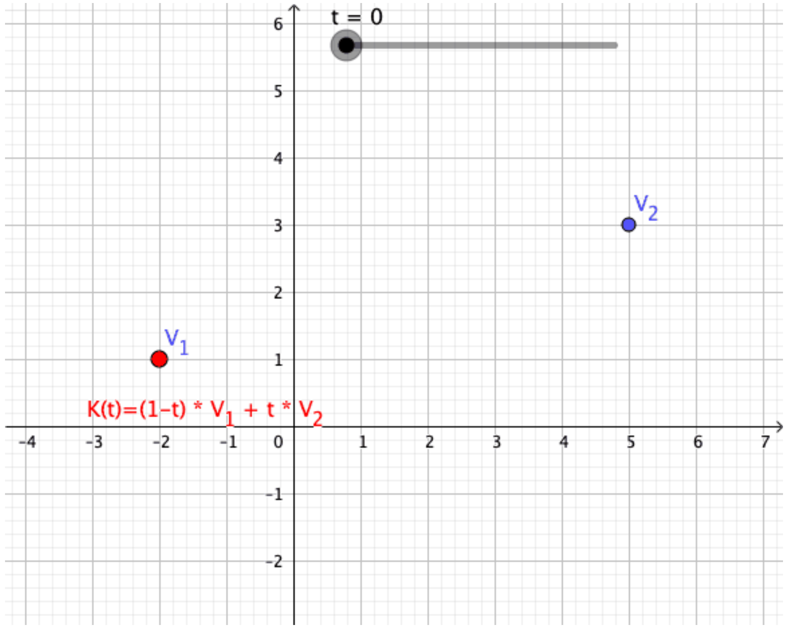
# Filippov's Convex Combination Method

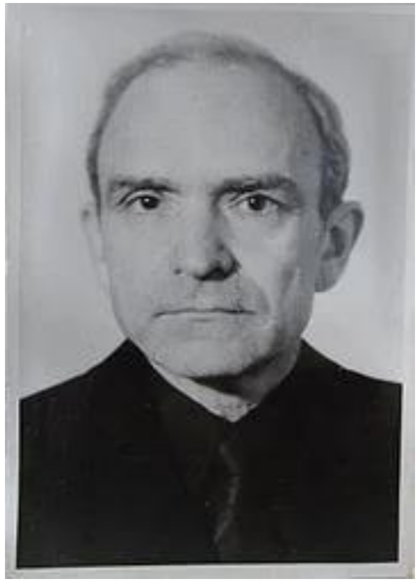
Convex combination of vectors $x_1, x_2, \ldots, x_n \in V$: $\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n$

where, $\alpha_i \in \mathbb{R}$ and $\alpha_1 + \alpha_2 + \ldots + \alpha_n = 1$

For two vectors $x_1$ and $x_2$, convex combination is of the form: $K = (1-t)x_1 + tx_2$

The vector K lies on the line joining $x_1$ and $x_2$ for any $t \in [0, 1]$



Aleksei Fedorovich Filippov

# Filippov's Convex Combination Method

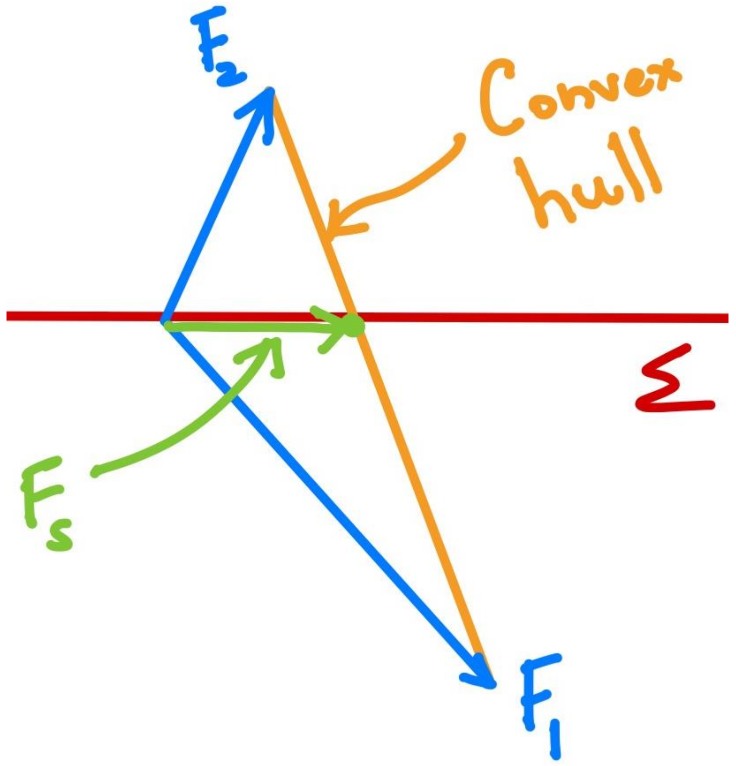
Filippov defined the sliding vector field as a convex combination of vector fields on either sides of $\Sigma$:

$$F_s = (1-\alpha)F_1 + \alpha F_2 \qquad 0 \le \alpha \le 1$$

The value of $\alpha$ is chosen such that $F_s$ is tangential to the discontinuous boundary $\Sigma$:

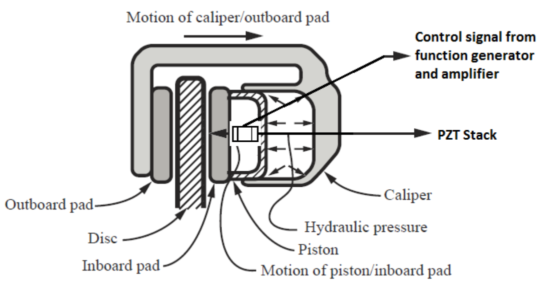
$$H_x\big((1-\alpha)F_1 + \alpha F_2\big) = 0$$

$$\Rightarrow \alpha = \frac{H_x F_1}{H_x(F_1 - F_2)}$$

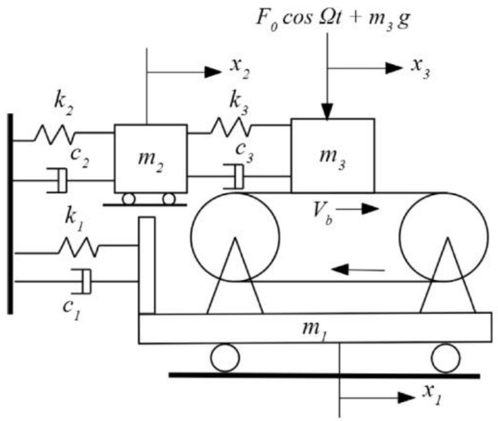


# Filippov Method - Applications

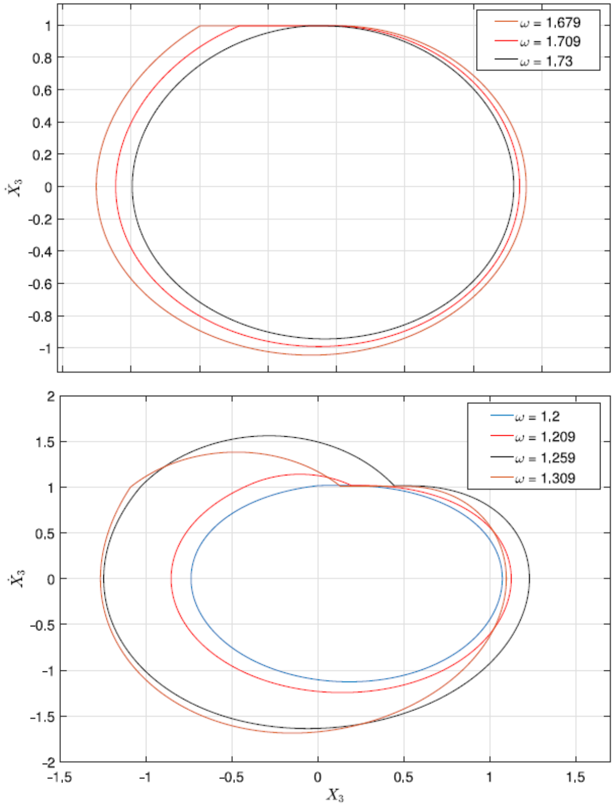
## Suppression of disc brake squeal




Application of Dither




3 DoF Disc Brake Model



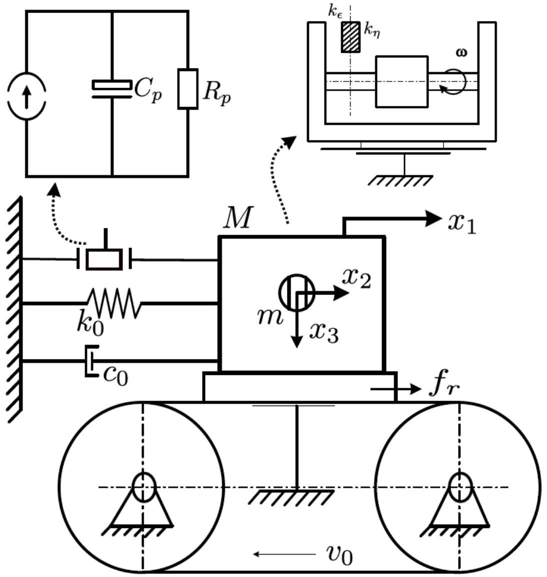

Modulation in the sliding region on the surface of discontinuity:

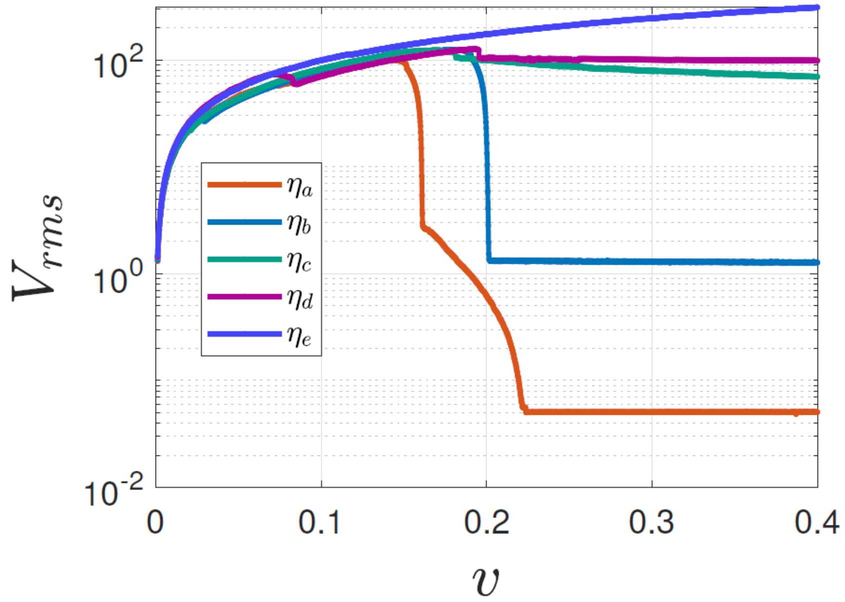
$$d = |\mu_s(\lambda_1 N + \lambda_2 F_0 cos\omega t)|$$


Bipin Balaram, B. Santhosh, Jan Awrejcewicz. *Frequency entrainment and suppression of stick-slip vibrations in a 3 DoF discontinuous disc brake model. Journal of Sound and Vibration, 538, 117224, 2022.*

# Filippov Method - Applications

## Energy harvesting from disc brake vibrations

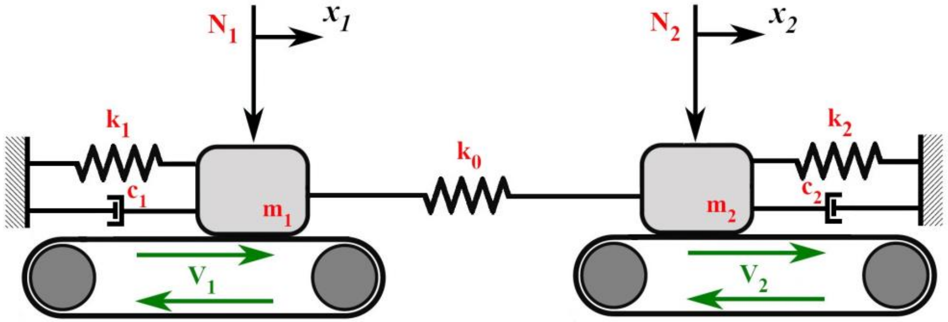



Godwin Sani, Bipin Balaram, Grzegorz Kudra, Jan Awrejcewicz. *Energy harvesting from friction-induced vibrations in vehicle braking systems in the presence of rotary unbalances. Energy, 289, 130007, 2024.*

# Higher Order Filippov Systems

Higher order sliding can occur in systems with more than one sliding surface

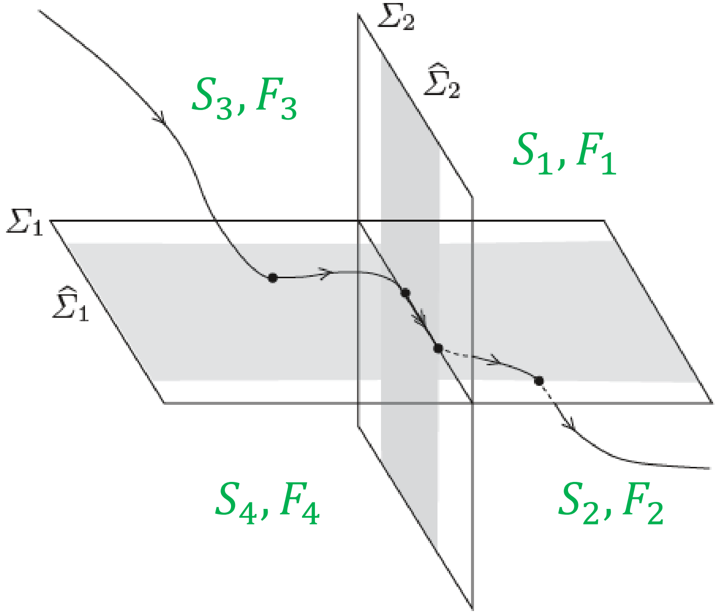



Coupled stick-slip oscillators

Bipin Balaram, B. Santhosh, Jan Awrejcewicz. *Synchronisation in Coupled Stick-slip Oscillators. (Under preparation).*

# Higher Order Sliding

The intersection of sliding surfaces $\Sigma_1$ and $\Sigma_2$ lead to higher order sliding motion




State space partitioned into 4 regions $S_1, S_2, S_3, S_4$ with vector fields $F_1, F_2, F_3, F_4$ respectively.

$$\dot{x} = \begin{cases} F_1(x), & \text{if } H_1(x) > 0,\ H_2(x) > 0, \\ F_2(x), & \text{if } H_1(x) > 0,\ H_2(x) < 0, \\ F_3(x), & \text{if } H_1(x) < 0,\ H_2(x) > 0, \\ F_4(x), & \text{if } H_1(x) < 0,\ H_2(x) < 0; \end{cases}$$

5 intersections:

$$\hat{\Sigma}_1^+ = S_1 \cap S_2$$
$$\hat{\Sigma}_1^- = S_3 \cap S_4$$
$$\hat{\Sigma}_2^+ = S_1 \cap S_3$$
$$\hat{\Sigma}_2^- = S_2 \cap S_4$$
$$\hat{S}_H = \hat{\Sigma}_1 \cap \hat{\Sigma}_2$$

# Generalised Filippov Method

Sliding vector field on $\hat{\Sigma}_1^-$:

$$(1-\alpha_1^-)F_3+\alpha_1^-F_4 \quad \text{where} \quad \alpha_1^-=\frac{H_{1,x}F_3}{H_{1,x}(F_3-F_4)}$$

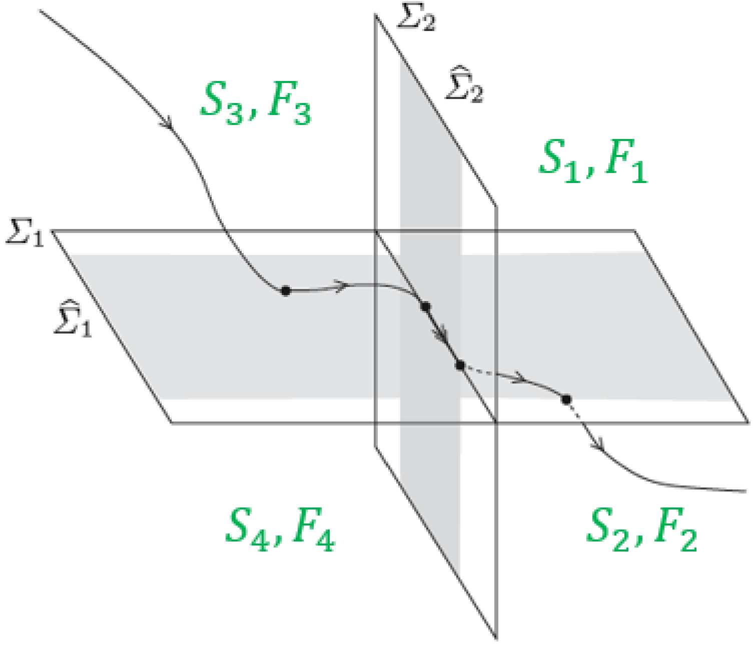
Vector fields on $\hat{\Sigma}_1^+$, $\hat{\Sigma}_2^+$ and $\hat{\Sigma}_2^-$ can be defined similarly.

Vector field on the higher order sliding region $\hat{S}_H$:

$$F_{SH}=\gamma_1F_1+\gamma_2F_2+\gamma_3F_3+\gamma_4F_4$$

$$\gamma_1+\gamma_2+\gamma_3+\gamma_4=1$$

$\gamma_i's$ are obtained from $H_{1,x}F_{SH}=0$ and $H_{2,x}F_{SH}=0$

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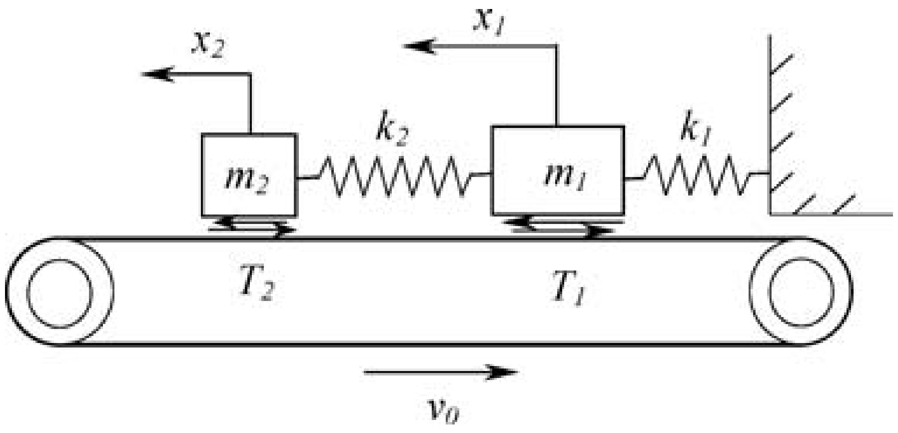
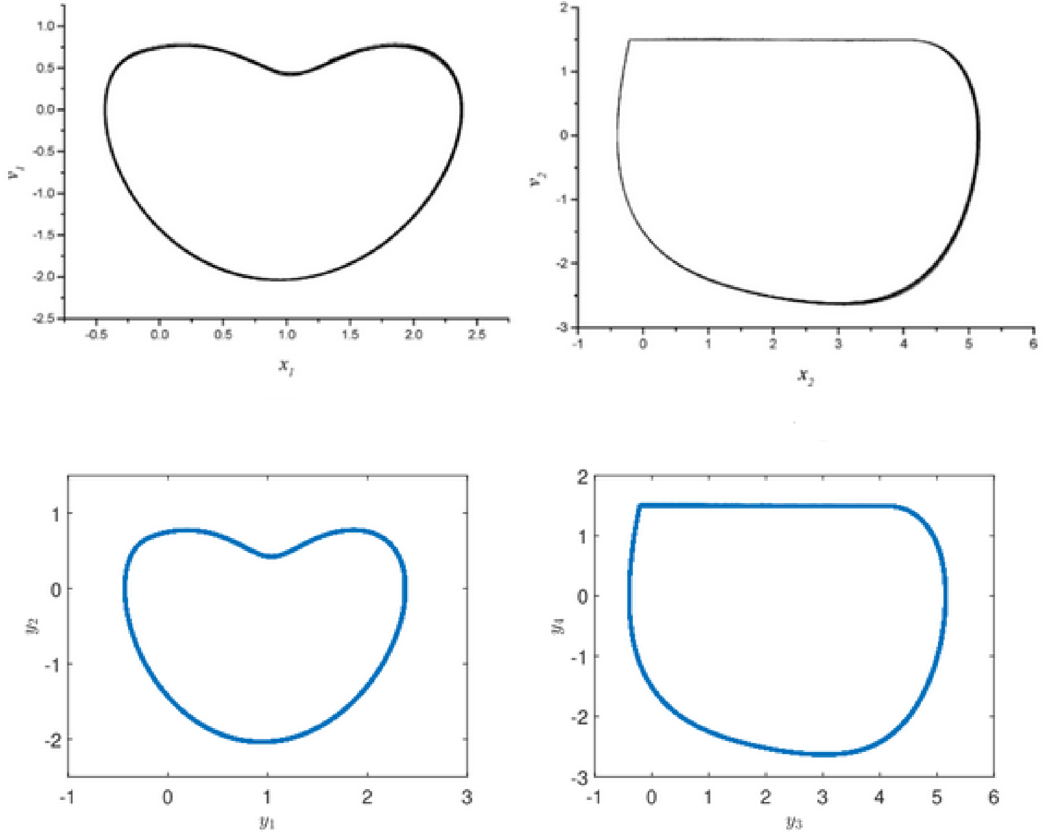
# Validation of the Method




J. Awrejcewicz, L. Dzyubak, C. Grebogi. *Estimation of Chaotic and Regular (Stick–Slip and Slip–Slip) Oscillations Exhibited by Coupled Oscillators with Dry Friction*. Nonlinear Dynamics (2005) 42: 383–394




Comparison for $v_0 = 1.5, T_{01} = 0.5, T_{02} = 19.62$

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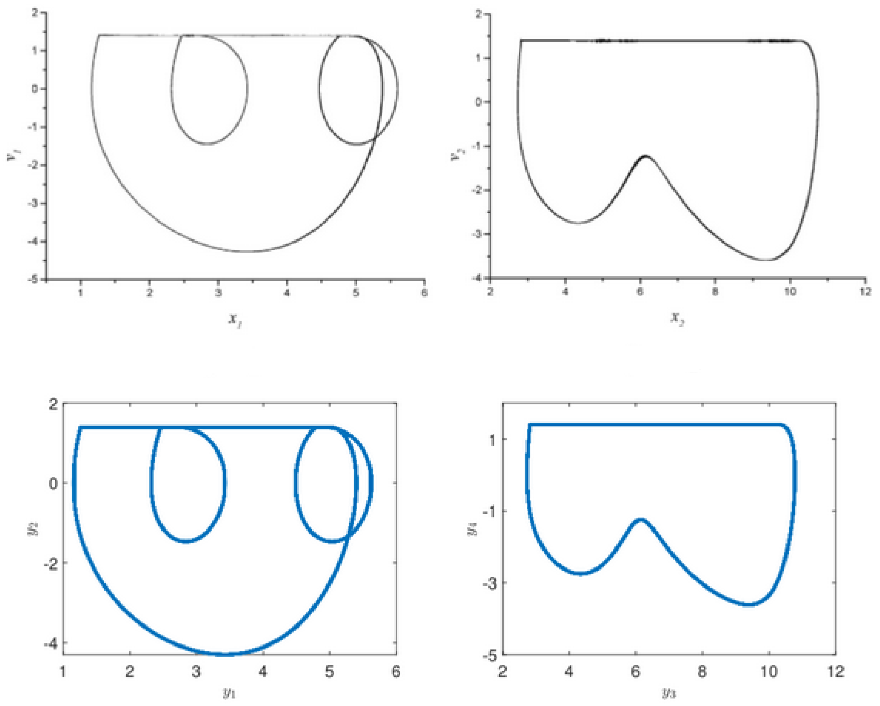
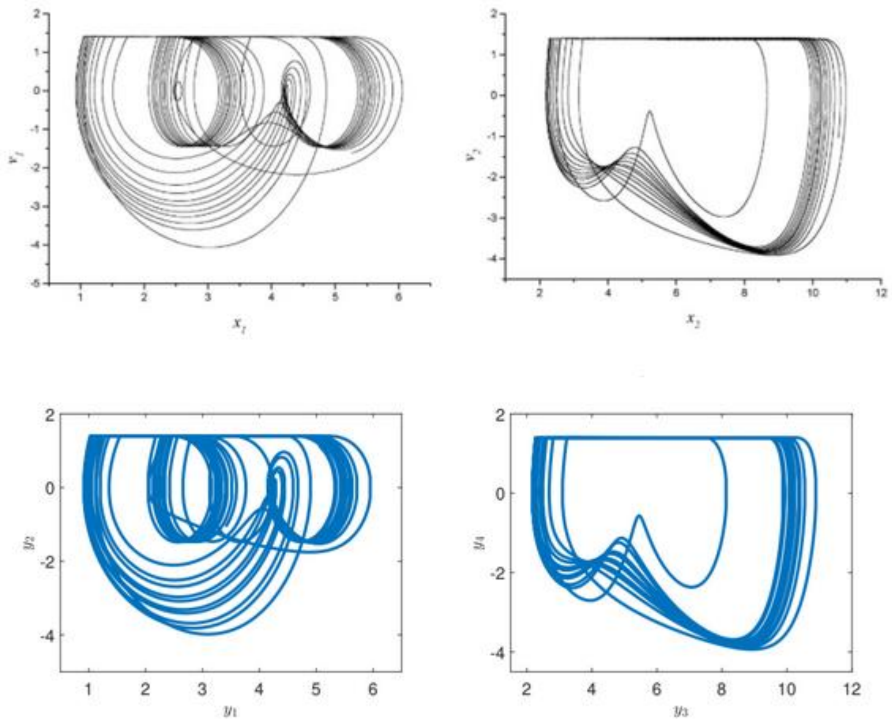
# Validation of the Method




Comparison for $v_0 = 1.4, T_{01} = 29.43, T_{02} = 40$




Comparison for $v_0 = 1.4, T_{01} = 29.43, T_{02} = 19.34$

Bipin Balaram, B. Santhosh, Jan Awrejcewicz. *Synchronisation in Coupled Stick-slip Oscillators. (Under preparation).*

# Work in Progress

- Higher order sliding orbits of coupled stick-slip oscillators
- Synchronisation properties of coupled stick-slip systems
- Grazing-sliding process in systems with two discontinuity surfaces
- Effect of higher order sliding on discontiuity induced bifurcations.

Dziękuję bardzo!