Dynamics of the plate with a complex planform

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Mathematical statement of the problem, based on the nonlocal theory

$$\sigma = \int_V K\left(|X' - X|, \tau\right) \sigma'(X') dX',$$

where σ, σ' are nonlocal and local stress tensors, $K(|X' - X|, \tau)$ is the nonlocal modulus, $\tau = e_0\alpha/l, \alpha$ is the internal characteristic length, e_0 is constant corresponded to material and l is external characteristic length

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} - \mu \nabla^2 \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{E_1}{1-\nu_1\nu_2} & \frac{\nu_1 E_2}{1-\nu_1\nu_2} & 0 \\ \frac{\nu_2 E_1}{1-\nu_1\nu_2} & \frac{E_2}{1-\nu_1\nu_2} & 0 \\ 0 & 0 & G \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{bmatrix},$$

where $\sigma = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix}$, $\mu = (e_0\alpha)^2$ is nonlocal parameter.

where E_1, E_2 are Young's modules, ν_1, ν_2 are Poisson's ratios, G is shear modulus.

$$D_{11}\frac{\partial^4 w}{\partial x^4} + 2(D_{12} + 2D_{66})\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w}{\partial y^4} = (1 - \mu\nabla^2)\left(-\rho\, h\, \frac{\partial^2 w}{\partial t^2}\right),$$

Boundary conditions:

1. Simply supported $w = 0, M_n = 0;$
2. Clamped $w = 0, \frac{\partial w}{\partial n} = 0.$

$$D_{11} = \frac{E_1 h^3}{12(1-\nu_1\nu_2)}, D_{22} = \frac{E_2 h^3}{12(1-\nu_1\nu_2)},$$

$$D_{12} = \nu_1 D_{22}, \quad D_{66} = \frac{Gh^3}{12}.$$

Variational statement of the problem

$$\Pi = \frac{1}{2}\iint_\Omega \left(D_{11}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + D_{22}\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2D_{12}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + 4D_{66}\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right) dxdy - \frac{\omega_L^2}{2}\iint_\Omega \left(w^2 + \mu\left(\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right)\right) dxdy$$

$$w = \sum_{i=1}^{n} c_i w_i(x, y),$$

where c_i are unknown coefficients, $w_i = g(x,y)\varphi_i$ are system of coordinate functions, whereas $g(x,y)$ is shape function depending on the boundary conditions and shape of the plate, φ_i is a complete system of functions, in particular, the set of power polynomials. Obviously, construction of the shape functions is quietly difficult problem when the shape of the plate is complicated, for example, contains cutouts.

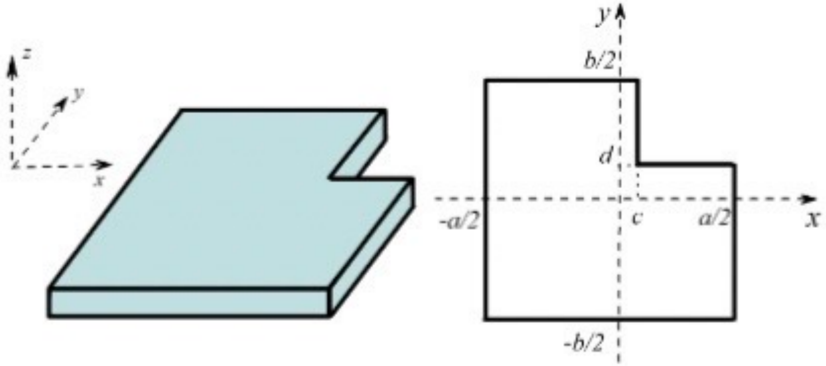
Finding unknown coefficients c_i is performed based on the condition of the minimum of the functional: $\frac{\partial \Pi}{\partial c_i} = 0, i = 1, n.$

$$(\{k_{ij}\} - \omega_L^2\{m_{ij}\})\{c_i\} = 0$$

$$k_{ij} = \iint_\Omega \left(D_{11}\frac{\partial^2 w_i}{\partial x^2}\frac{\partial^2 w_j}{\partial x^2} + D_{22}\frac{\partial^2 w_i}{\partial y^2}\frac{\partial^2 w_j}{\partial y^2} + D_{12}\left(\frac{\partial^2 w_i}{\partial x^2}\frac{\partial^2 w_j}{\partial y^2} + \frac{\partial^2 w_i}{\partial x^2}\frac{\partial^2 w_j}{\partial y^2}\right) + 4D_{66}\frac{\partial^2 w_i}{\partial x \partial y}\frac{\partial^2 w_j}{\partial x \partial y}\right) dxdy$$

$$m_{ij} = \rho h \iint_\Omega \left(w_i w_j + \mu\left(\frac{\partial w_i}{\partial x}\frac{\partial w_j}{\partial x} + \frac{\partial w_i}{\partial y}\frac{\partial w_j}{\partial y}\right)\right) dxdy, \qquad i, j = 1..n.$$

R-functions method for investigation of plates with complex shape



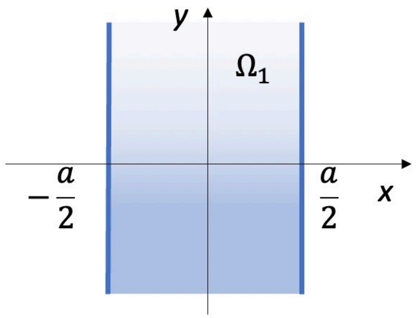
$$\Omega = \Omega_1 \wedge \Omega_2 \wedge (\Omega_3 \vee \Omega_4)$$

The problem of construction of the basis functions for the Ritz method requires consideration of shape information.

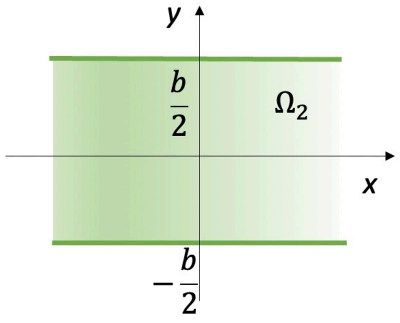
For example for rectangular plate $g(x,y) = \left(\frac{a}{2} - x\right)^{r_1}\left(\frac{a}{2} + x\right)^{r_2}\left(\frac{b}{2} - y\right)^{r_3}\left(\frac{b}{2} + y\right)^{r_4}$.

For this purpose, we propose to use the R- functions method , which describes the geometry of the plate analytically.

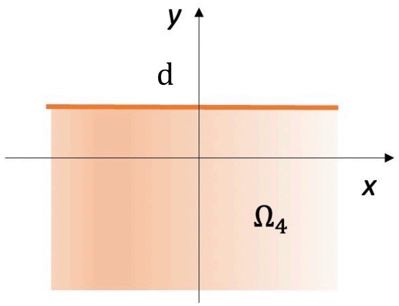
In order to construct the equation of the domain boundary, it is needed to write the characteristic function of the domain consisting of the characteristic functions Ω_i of its subdomains.



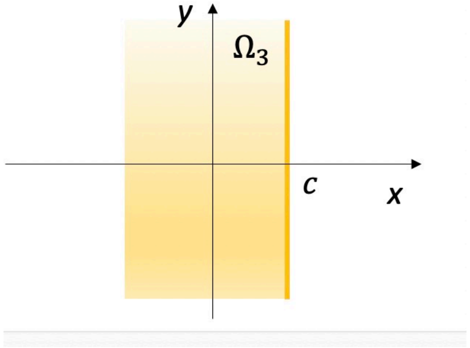
$$f_1 = \frac{1}{a}\left(\left(\frac{a}{2}\right)^2 - x^2\right) \geq 0$$



$$f_2 = \frac{1}{b}\left(\left(\frac{b}{2}\right)^2 - y^2\right) \geq 0,$$



$$f_4 = d - y \geq 0$$



$$f_3 = c - x \geq 0,$$

According to the theorem proved by V.L. Rvachev it is sufficient to perform a formal replacement of the Ω_i by corresponded continuous functions f_i and Boolean operators $\wedge, \vee, -$ by the corresponding R-functions to obtain the equation $\omega(x, y) = 0$ of the boundary of the domain.

We apply the following R-functions system

$$x \vee_0 y = x + y + \sqrt{x^2 + y^2}, (R - disjunction),$$
$$x \wedge_0 y = x + y - \sqrt{x^2 + y^2}, (R - conjunction),$$
$$\overline{x} = -x, (R - negation).$$

$$\omega = f_1 \wedge_0 f_2 \wedge_0 (f_3 \vee_0 f_4).$$

$$\omega = \frac{1}{a}\left(\left(\frac{a}{2}\right)^2 - x^2\right) \wedge_0 \left(\left(\frac{b}{2}\right)^2 - y^2\right) \wedge_0 \left((d - y) \vee_0 (c - x)\right)$$

The logical predicate of the domain is constructed as follows:

$$\Omega = (\Omega_1 \wedge_0 \Omega_2) \wedge_0 (\Omega_3 \vee_0 \Omega_4)$$

According to the algorithm mentioned above, one can obtain the equation of the boundary domain as

$$\omega = (f_1 \wedge_0 f_2) \wedge_0 (f_3 \vee_0 f_4).$$

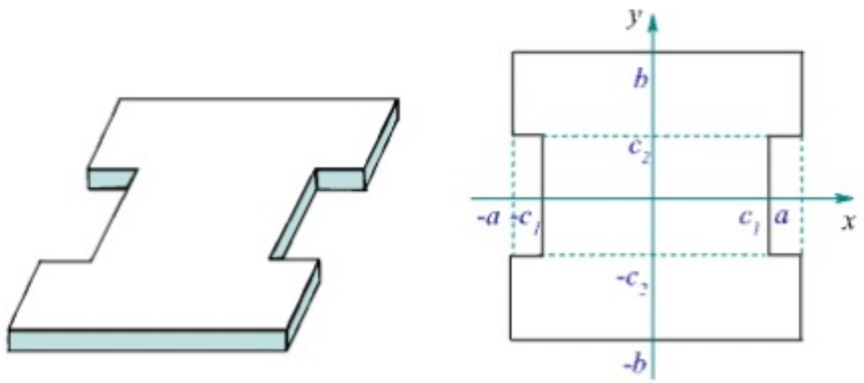
where

$$f_1 = \frac{1}{2a}\left(a^2 - x^2\right) \geq 0, f_2 = \frac{1}{2b}\left(b^2 - y^2\right) \geq 0$$

are the region between $x = \pm a$ and the region between $y = \pm b$, whereas

$$f_3 = \frac{1}{2c_1}\left(c_1^2 - x^2\right) \geq 0, f_4 = \frac{1}{2c_2}\left(y^2 - c_2^2\right) \geq 0$$

are the region between $x = \pm c_1$ and the region outside the lines $y = \pm c_2$.



System of basis functions $w_i = g(x,y)\varphi_i$, φ_i is a complete system of functions.

We use power polinomials: $1, x, y, x^2, xy, y^2, ..$

Structure of solutions:

1. Simply supported boundary conditions: $w = \omega P,$ $P = \sum_{i=1}^{n} c_i \varphi_i,$ $w = \sum_{i=1}^{n} c_i \omega \varphi_i.$

basis functions: $w_1 = \omega, w_2 = \omega x, w_3 = \omega y, w_4 = \omega x^2, w_5 = \omega xy,\ w_6 = \omega y^2, ..$

2. Clamped boundary conditions: $w = \omega^2 P.$

basis functions: $w_1 = \omega^2, w_2 = \omega^2 x, w_3 = \omega^2 y, w_4 = \omega^2 x^2, w_5 = \omega^2 xy,\ w_6 = \omega^2 y^2, ..$

3. Mixed boundary conditions: $w = \omega_1 \omega P.$

If part of the plate boundary is clamped (ω_1) and the remaining part is simply supported.

basis functions: $w_1 = \omega_1 \omega, w_2 = \omega_1 \omega x, w_3 = \omega_1 \omega y, w_4 = \omega_1 \omega x^2, w_5 = \omega_1 \omega xy,\ w_6 = \omega_1 \omega y^2, ..$

Some validation problems

Validation of the proposed method has been performed by solving several testing problems and comparison obtained results with existing ones.

The first case study describes the linear vibrations of isotropic square nanoplate with simply supported boundary conditions.

$$E = E_1 = E_2, \nu = \nu_1 = \nu_2, G = E/(2(1+\nu)). \qquad E = 30MPa, \rho = 1220kg/m^3, \nu = 0.3$$

The dimensionless frequency parameter $\overline{\omega} = \omega\, h \sqrt{\frac{\rho}{G}}$ calculated for various values of the nonlocal parameter μ is presented in Table 1,

$2a = 10nm, \frac{a}{h} = 10.$

Table 1: Dimensionless linear frequency ϖ of the isotropic simply supported square plate

μ	0	1	2	3	4	5
[14]	0.0963	0.0880	0.0816	0.0763	0.0720	0.0683
RFM	0.0963	0.0881	0.08158	0.0764	0.0720	0.0684

Table 2: Dimensionless linear frequency ϖ of the orthotropic clamped square plate

μ	0	1	2	3	4
[29] DOM	10.5941	9.5446	8.7526	8.1267	7.617
[29] FEM	10.5533	9.5125	8.7242	8.1016	7.5949
[19]	10.5941	9.54546	8.7525	8.1267	7.617
RFM	10.5898	9.5422	8.7496	8.1241	7.61463

Further, we consider the problem studied in [19,29], namely the clamped orthotropic graphene sheet with the following mechanical parameters:

$$E_1 = 1765GPa, E_2 = 1588GPa, G = 678.85GPa, \rho = 2300kg/m^3, \nu_1 = 0.3, \nu_2 = 0.27,$$

and geometrical parameters: $2a = 10.2nm, h = 0.34nm, \frac{b}{a} = 1$ has been investigated.

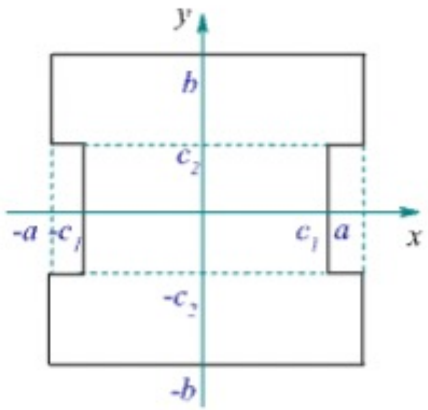
The dimensionless frequencies $\overline{\omega} = \frac{(2a)^2\omega}{h}\sqrt{\frac{\rho}{E_1}}$ are provided for nonlocal parameter μ (in the range 0..4 nm^2), see Table 2.

[14] R. Aghababaei, J. N. Reddy, Nonlocal third-order shear deformation plate theory with application to bending and vibration of plates, Journal of Sound and Vibration 326 (1-2) (2009) 277–289. doi:10.1016/j.jsv.2009.04.044.

[29] J. K. Phadikar, S. C. Pradhan, Variational formulation and finite element analysis for nonlocal elastic nanobeams and nanoplates, Computational Materials Science 49 (3) (2010) 492–499. doi:10.1016/j.commatsci.2010.05.040.

[19] A. R. Shahidi, S. Shahidi, A. Anjomshoae, E. Raeisi Estabragh, Vibration analysis of orthotropic triangular nanoplates using nonlocal elasticity theory and Galerkin method, Journal of solid mechanics 8 (3) (2016) 679–692. URL https://www.sid.ir/en/journal/ViewPaper.aspx?id=511710

To validate the proposed approach for plate with the complicated shape, the ABAQUS was used for isotropic plate with mechanical properties.



$$E = 30MPa, \rho = 1220kg/m^3, \nu = 0.3$$

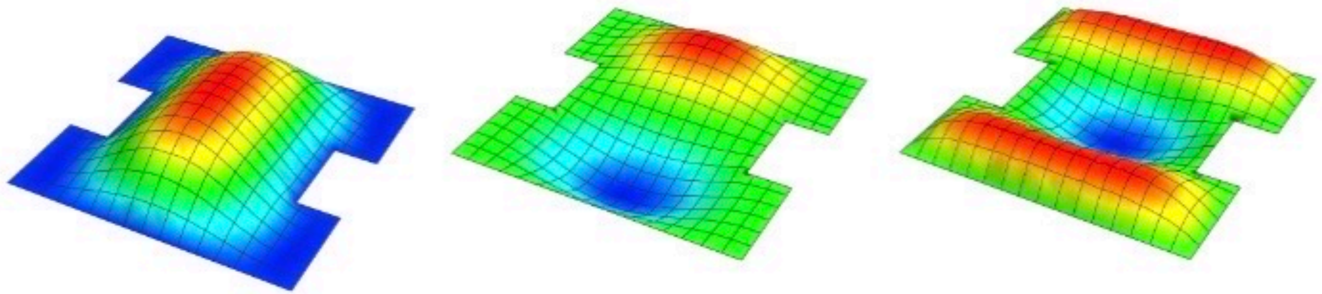
For this case study the classical theory μ=0 was considered. The plate is assumed to be simply supported or clamped. In the Table 3 the results of our calculation by method based on R-function theory are presented as well.

Table 3: Dimensionless linear frequency $\overline{\omega} = \frac{(2a)^2\omega}{h}\sqrt{\frac{\rho}{E_1}}$ of the isotropic plate with two outer cutouts ($\mu = 0, \nu = 0.3$)

	$c_1/a = 0.7, c_2/a = 0.3$	
	simply supported	clamped
RFM	11.712	16.707
ABAQUS	11.562	16.730
	$c_1/a = 0.6, c_2/a = 0.4$	
	simply supported	clamped
RFM	14.041	21.480
ABAQUS	14.090	21.547

Numerical study

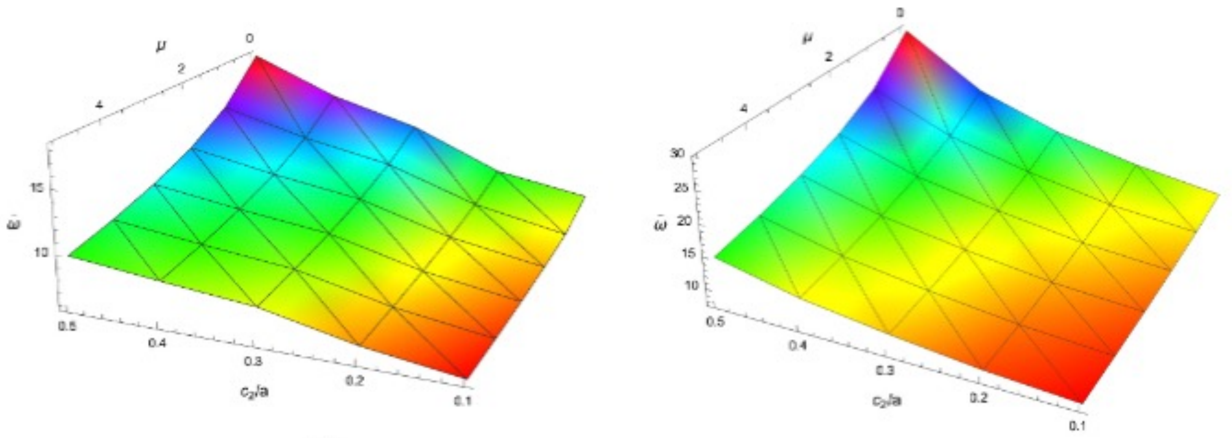
Further numerical calculations are performed for small-scale plate with the following geometrical parameters $a = 5nm, h/2a = 0.1, b/a = 1$ and the plate is supposed to be from the isotropic material. The presented results contain the first three vibration modes calculated for $\mu = 1$ nm^2 and size of cutouts associated with following ratio $\frac{c_1}{a} = 0.7, \frac{c_2}{a} = 0.3$, the corresponding frequency parameters $\overline{\omega} = \frac{(2a)^2 \omega_i}{h} \sqrt{\frac{\rho}{E_1}}, i = 1,2,3$ are shown as well.



(a) $\varpi_1 = 11.042$ (b) $\varpi_2 = 15.391$ (c) $\varpi_3 = 22.123$

Figure 2: Vibration modes associated with the first, second and third vibration frequencies.

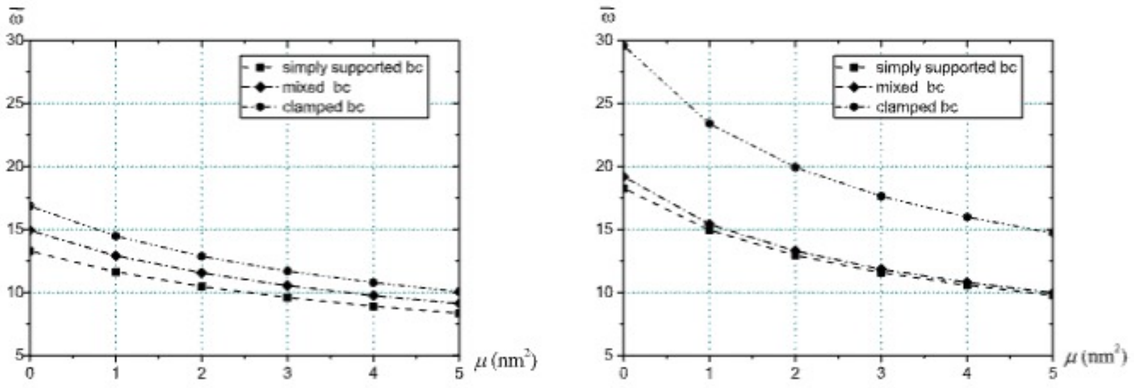
Next figure demonstrates the results of studying the effect of the size of the cutouts on the frequency parameter $\overline{\omega} = \frac{(2a)^2\omega}{h}\sqrt{\frac{\rho}{E_1}}$ depending on the change in the nonlocal parameter. It is assumed that $\frac{c_1}{a} + \frac{c_2}{a} = 1$.



(a) (b)

Figure 3: Dimensionless frequencies $\bar{\omega}$ in terms of cutout size ratio c_2/a and nonlocal parameter μ for isotropic nanoplate; (a) simply supported, (b) clamped.

The influence of the boundary conditions was investigated for two cutout sizes, $\frac{c_1}{a}=0.7, \frac{c_2}{a} = 0.3$ and. $\frac{c_1}{a}=0.5, \frac{c_2}{a} = 0.5$. Here we considered the above-mentioned boundary conditions, where the plate is simply supported or clamped along the entire edge, as well as mixed conditions with clamped part of the boundary $\omega_2 = f_2$ and remaining simply supported part of the edge.



(a) (b)

Figure 4: Dimensionless frequencies $\bar{\omega}$ for three types of boundary conditions versus nonlocal parameter μ; (a) $c_1/a = 0.7; c_2/a = 0.3$, (b) $c_1/a = 0.5, c_2/a = 0.5$.

Next, we consider an orthotropic nanoplate (mechanical properties mentioned above) with a complex shape and geometrical ratios $2a = 10nm, h = 1nm, \frac{b}{a} = 1, \frac{c_1}{a} + \frac{c_2}{a} = 1.$ By varying the values of the nonlocal parameter μ and size of the cutouts, the frequency parameter

$$\overline{\omega} = \frac{(2a)^2\omega}{h}\sqrt{\frac{\rho}{E_1}}$$ was calculated.

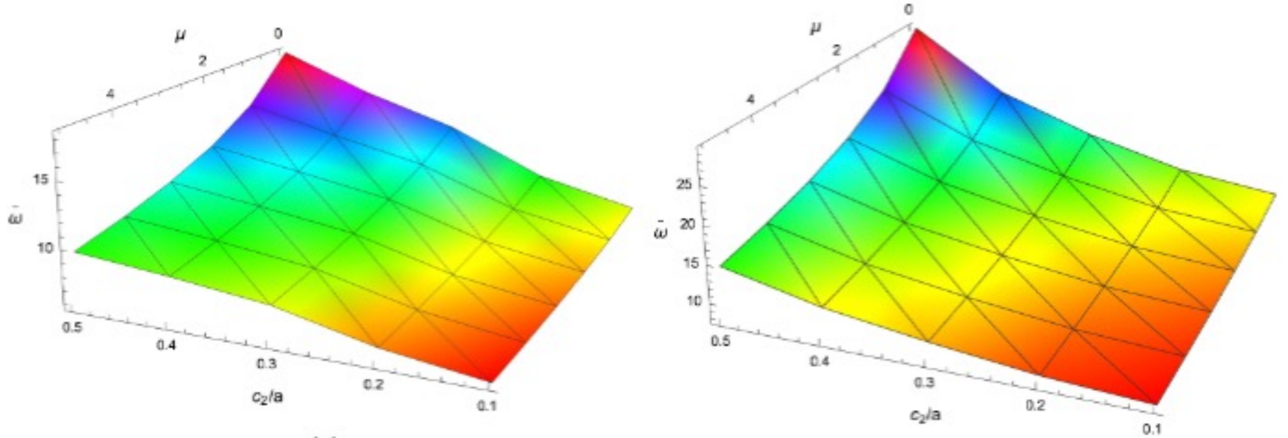


Figure 5: Dimensionless frequencies $\bar{\omega}$ in terms of cutout size ratio c_2/a and nonlocal parameter μ for orthotropic nanoplate; (a) simply supported, (b) clamped.

Figure 6 shows the values of the frequency parameter $\overline{\omega}$ for isotropic and orthotropic plates in the case of simply supported and clamped nanoplates. Geometrical parameters here are taken as $\frac{c_1}{a}$ =0.5, $\frac{c_2}{a}$ =0.5.

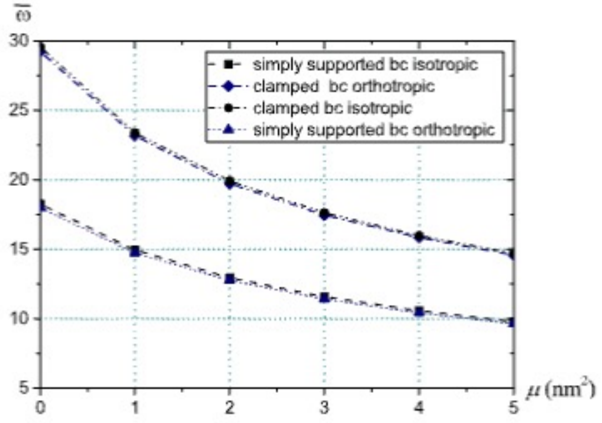


Figure 6: Dimensionless frequencies $\bar{\omega}$ for different boundary conditions versus nonlocal parameter μ; $c_1/a = 0.5, c_2/a = 0.5$.

Concluding remarks

The presented work is devoted to the study of vibrations of orthotropic nano/microplates of complex shapes. The governing equations are based on the Kirchoff-Love hypothesis, and the nonlocal theory is used to take into account small-scale effects. The proposed approach uses a variational statement in combination with the Ritz method. At the same time, the approach to construct a coordinate system based on the R-functions theory is fundamentally new for this class of problems, and it allows one to study plates of various geometric shapes, as well as the influence of different types of boundary conditions. A numerical simulation was performed for isotropic and orthotropic square nanoplate with cutouts on opposite sides. Moreover, we considered three types of boundary conditions, including mixed ones. The developed approach allowed us to observe that an increase in the nonlocal parameter decreases the value of the frequency parameter for a complex shape plate for all types of boundary conditions considered, while the small-scale effect is more pronounced for large plate cutouts and clamped boundary.