Lodz University of Technology

Department of Automation, Biomechanics and Mechatronics

Approximate Analytical Solution of a 4-DOF Variable-Length Pendulum Model using the multiple scale approach

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Plan of presentation

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Introduction

- Analytical solutions are powerful tools in various fields of science and engineering, allowing researchers to obtain explicit mathematical expressions that describe the behavior of complex systems. However, there are situations where traditional analytical methods may fail to capture the intricacies of a system accurately. This is particularly true when dealing with systems that exhibit complex nonlinear features, where phenomena occur at significantly different spatial or temporal scales.

- To overcome this challenge, multiple scale techniques are employed to develop analytical solutions that account for the interactions and dynamics occurring across different scales. These techniques aim to capture the essential features of the system by breaking it down into different levels of detail and deriving approximate solutions for each scale. By integrating these solutions, a comprehensive description of the system's behavior can be obtained.

- Multiple scale techniques often involve the application of perturbation theory, which assumes that the system can be expressed as a small perturbation from a simpler, well-understood base state. Perturbation methods, such as the method of matched asymptotic expansions, homogenization, or averaging, are then employed to derive solutions that incorporate the effects of both the dominant scales and the perturbations.

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Introduction

Approximate solutions finding techniques

1. Pertubation method
2. Variation of parameters
3. Homotopy analysis method (HAM)
4. Numerical method
5. Approximation Techniques
   - Adomian decomposition method
   - variational iteration method, and
   - multiple scale method

Why is the analytical solution of these equations significant?

1. Insight and Understanding
2. Efficiency and Accuracy
3. Model Validation
4. Design and Optimization
5. Theoretical Advancements

The multiple scale technique Process

1. Start with the second-order nonlinear ODE in standard form: $\frac{d^2y}{dt^2} + \varepsilon f\left(t, y, \frac{dy}{dt}\right) = 0$ ⋯ (1)

2. Assume a solution of the form: $y(t) = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \cdots$ ⋯ (2)

3. Substitute the assumed solution into the ODE and collect terms according to powers of ε. Equate each term to zero to obtain a series of equations at different orders of ε.

4. At the leading order (ε^0), we will obtain a linear homogeneous ODE:

$$\frac{d^2y_0}{dt^2} + f\left(t, y_0, \frac{dy_0}{dt}\right) = 0 \quad \cdots (3)$$

Solve this equation to find the leading-order solution $y_0(t)$.

5. At the first order $(\boldsymbol{\varepsilon}^{1})$, we will obtain a linear inhomogeneous ODE:

$$\frac{d^2y_1}{dt^2} + f_t\left(t, y_0, \frac{dy_0}{dt}\right) y_1 + f_n\left(t, y_0, \frac{dy_0}{dt}\right) = 0 \qquad \cdots (4)$$

Solve this equation to find the first-order correction $\boldsymbol{y_1(t)}$.

6. Continue this process for higher orders of $\boldsymbol{\varepsilon}$ if necessary.

7. The final approximate solution will be the sum of the leading-order solution and all the corrections:

$$y(t) = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \cdots \qquad \cdots (5)$$

we can truncate the series at any desired order based on the accuracy required.

8. Solve for the coefficients in each correction term by using appropriate boundary or initial conditions, depending on the problem's nature.

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The variable-length pendulum system/Swinging Atwood's Machine (SAM)

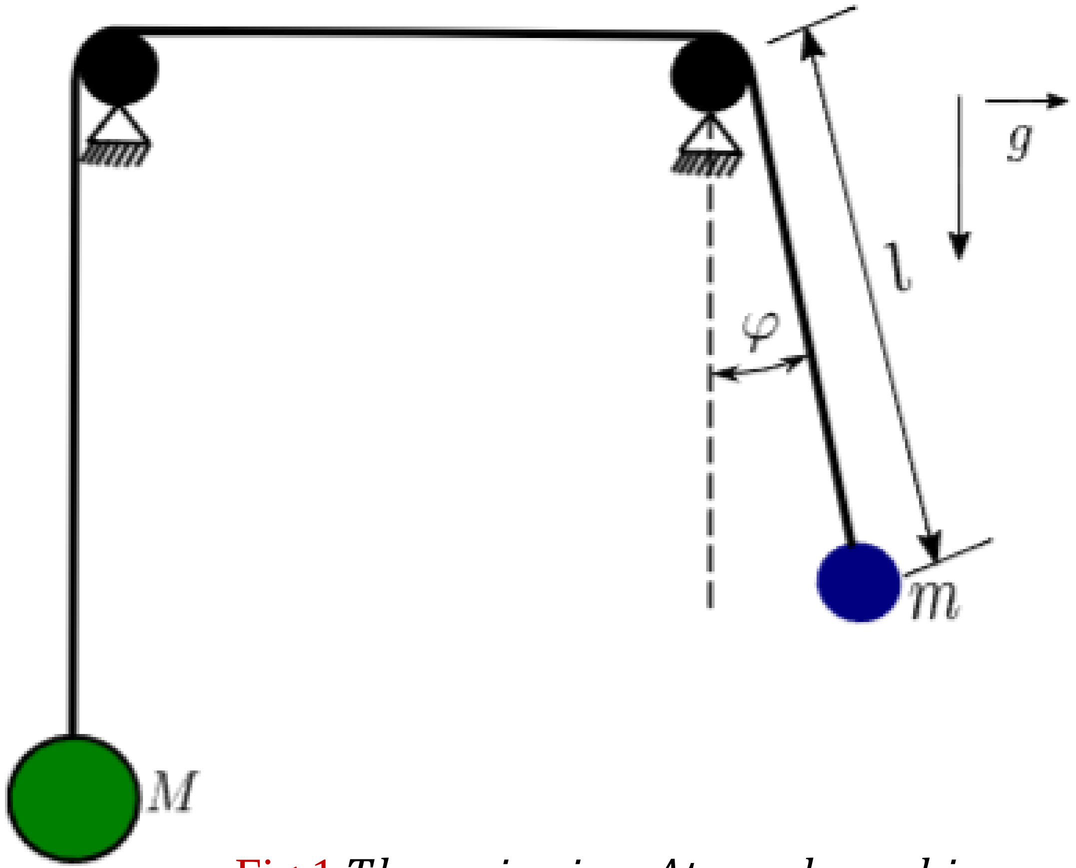

Fig.1 *The swinging Atwood machine*

$$l\ddot{\varphi} + 2\dot{l}\dot{\varphi} + g\sin\varphi = 0$$

$$ml(t)\dot{\varphi}^2(t) - Mg + mg\cos\varphi(t) = (M+m)\ddot{l}(t), \qquad \cdots (6)$$

$$(\mu_m + 1)\ddot{l}(t) - l(t)\dot{\varphi}(t) + g(\mu_m - \cos\varphi(t)),$$

$$\mu_m = \frac{M}{m}, \quad \varphi = \frac{\pi}{2}$$

$$\omega_2^2 = \frac{g}{l},\ \omega_4^2 = \frac{\omega_2^2}{\omega_1^2},\ \sigma_1 = \frac{\lambda^3}{(\mu+1)\omega_1^2} + \frac{\mu\omega_4^2}{\mu+1} - \frac{\omega_4^2}{\mu+1},\ \sigma_2 = \frac{\lambda^2}{(\mu+1)\omega_1^2},\ \sigma_3 = \frac{\omega_4^2}{3(\mu+1)},$$

$$\sigma_4 = \frac{2\lambda^2}{(\mu+1)\omega_1},\ \sigma_5 = \frac{2\lambda}{(\mu+1)\omega_1},\ \sigma_6 = -\frac{\lambda}{\mu+1}, \qquad \cdots (7)$$

$$\sigma_7 = \frac{1}{\mu+1},\ \zeta_1 = \omega_4^2,\ \zeta_2 = \frac{\omega_4^2}{6},\ \zeta_3 = \frac{2}{\omega_1},$$

$$\zeta_4 = \frac{2}{\lambda},\ \zeta_5 = \frac{1}{\lambda}.$$

$$\sin\phi(t) = \left(\phi(t) - \frac{(\phi(t))^3}{6}\right),$$

$$\cos\phi(t) = \left(1 - \frac{(\phi(t))^2}{2}\right).$$

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Final dimensionless form

$$\sigma_1 + \sigma_2 x(\tau) + \sigma_3 \phi(\tau)^2 + \sigma_4 \dot{\phi}_1(\tau) + \sigma_5 x(\tau)\dot{\phi}(\tau) + \sigma_6 \dot{\phi}(\tau)^2 + \sigma_7 x(\tau)\dot{\phi}(\tau)^2 + \ddot{x}(\tau) = 0, \quad \cdots (8)$$

$$\zeta_1 \phi(\tau) - \zeta_2 \phi(\tau)^3 + \zeta_3 \dot{x}(\tau) + \zeta_4 \dot{\phi}(\tau)\dot{x}(\tau) + \ddot{\phi}(\tau) + \zeta_5 x(\tau)\ddot{\phi}(\tau) = 0. \quad \cdots (9)$$

Multiple scale approach technique

The analysis focused on a localized region near the system's static equilibrium. A small parameter represented as $0 < \varepsilon \ll 1$ is introduced to characterize the amplitudes of the oscillations within this region.

This parameter allows us to establish the following relationship:

$$x(\tau) = \varepsilon\alpha(\tau : \varepsilon), \quad \phi 1(\tau) = \varepsilon\gamma(\tau : \varepsilon) \quad \cdots (10)$$

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Multiple scale approach technique

This allows us to consider the following approximation:

$$\begin{aligned} x(\tau) &= \varepsilon\alpha\left(\tau_0, \tau_1\right), \ \phi(\tau) = \varepsilon\gamma\left(\tau_0, \tau_1\right), \\ \dot{x}(\tau) &= \varepsilon\alpha'\left(\tau_0, \tau_1\right), \dot{\phi}(\tau) = \varepsilon\gamma'\left(\tau_0, \tau_1\right), \\ \ddot{x}(\tau) &= \varepsilon\alpha''\left(\tau_0, \tau_1\right), \ \ddot{\phi}(\tau) = \varepsilon\gamma''\left(\tau_0, \tau_1\right), \\ \tau &= \tau_0, \ \sigma_1 = \varepsilon^2\tilde{\sigma_1}, \ \sigma_4 = \varepsilon^1\tilde{\sigma_4}, \ \sigma_7 = \varepsilon^{-1}\tilde{\sigma_7}, \\ \zeta_2 &= \varepsilon^{-1}\tilde{\zeta_2}, \ \zeta_3 = \varepsilon\tilde{\zeta_3}, \ \zeta_5 = \varepsilon^0\tilde{\zeta_5}. \end{aligned} \quad \cdots (11)$$

The time-dependent variable $\chi(\tau)$ and $\phi(\tau)$ can be considered as a power series of ε

$$\begin{aligned} x(\tau) &= \sum_{k=1}^{2} \varepsilon^k x, k\left(\tau_0, \ \tau_1\right) + O\left(\varepsilon^k\right), \\ \phi(\tau) &= \sum_{k=1}^{2} \varepsilon^k \phi, k\left(\tau_0, \ \tau_1\right) + O\left(\varepsilon^k\right), \end{aligned} \quad \cdots (12)$$

The time scales are represented by $\tau_n = \varepsilon^n\tau \ (n = 0, 1)$, τ_0 – faster; τ_1 – slowest

Multiple scale approach technique

To convert derivatives with respect to τ to the new time scales τ_n, the following operators are employed:

$$\frac{d}{d\tau} = \frac{\partial}{\partial\tau_0} + \varepsilon\frac{\partial}{\partial\tau_0}, \quad \frac{d^2}{d\tau^2} = \frac{\partial^2}{\partial\tau_0^2} + 2\varepsilon\frac{\partial^2}{\partial\tau_0\partial\tau_1} + O\left(\varepsilon^2\right) \qquad \cdots (13)$$

The operators neglect terms of $O(\varepsilon^2)$ and higher.
To obtain the PDE groups corresponding to different powers of ε, we substitute equ. (10)-(13) into the dimensionless form of the governing equations. This procedure leads to the derivation of the preceding four linear PDEs. Based on the perturbation parameter $\boldsymbol{\varepsilon}$, the splitting method is employed for obtaining these PDEs. These equations are the orders of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon^2}$.

(i) First-order equations (coefficient "1" at ε^1)

$$\frac{\partial^2\alpha_1}{\partial\tau_0^2} + \sigma_2\alpha_1 = 0,$$

$$\frac{\partial^2\gamma_1}{\partial\tau_0^2} + \varsigma_1\gamma_1 = 0. \qquad \cdots (14)$$

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Multiple scale approach technique…

(ii) Second-order equations (coefficient “2" at ε^2)

$$\widetilde{\sigma}_1 + \sigma_2\alpha_2 + \sigma_3{\gamma_1}^2 + \widetilde{\sigma}_4\gamma_1\frac{\partial\gamma_1}{\partial\tau_0} + \sigma_5\alpha_1\gamma_1\frac{\partial\gamma_1}{\partial\tau_0} + \sigma_6\left(\frac{\partial\gamma_1}{\partial\tau_0}\right)^2 + \widetilde{\sigma}_7\alpha_1\left(\frac{\partial\gamma_1}{\partial\tau_0}\right)^2 + 2\frac{\partial^2\alpha_1}{\partial\tau_0\partial\tau_1} + \frac{\partial^2\alpha_2}{\partial\tau_0^2} = 0, \quad \cdots(15)$$

$$\zeta_1\gamma_2 - \widetilde{\zeta}_2\gamma_1^3 + \widetilde{\zeta}_3\frac{\partial\alpha_1}{\partial\tau_0} + \zeta_4\frac{\partial\alpha_1}{\partial\tau_0}\frac{\partial\gamma_1}{\partial\tau_0} + 2\frac{\partial^2\gamma_1}{\partial\tau_0\partial\tau_1} + \widetilde{\zeta}_5\alpha_1\frac{\partial^2\gamma_1}{\partial\tau_0^2} + \frac{\partial^2\gamma_2}{\partial\tau_0^2} = 0, \quad \cdots(16)$$

The resulting established solutions of Equ. (14) are presented as follows:

$$\alpha_1 = e^{i\sigma_2\tau_0}B_1(\tau_1) + e^{-i\sigma_2\tau_0}\widetilde{B}_1(\tau_1),$$
$$\gamma_1 = e^{i\zeta_1\tau_0}B_3(\tau_1) + e^{-i\zeta_1\tau_0}\widetilde{B}_3(\tau_1). \quad \cdots(17)$$

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$$\alpha_2 = \frac{i\sigma_5 B_1(\tau_1) B_2(\tau_1) e^{i\tau_0(\zeta_1+\sigma_2)}}{\zeta - 2 + 2\sigma_2} - \frac{-2\zeta_1^2\sigma_6 B_2(\tau_1)\tilde{B_2}(\tau_1) - 2\sigma_3 B_2(\tau_1)\tilde{B2}(\tau_1) + \tilde{\sigma_1}}{\sigma_2{}^2} - \frac{i\sigma_5\tilde{B_1}(\tau_1)\tilde{B_2}(\tau_1) e^{-i\tau_0(\zeta_1+\sigma_2)}}{\zeta_1 + 2\sigma_2} - \frac{e^{2i\zeta_1\tau_0}\left(\zeta_1^2\sigma_6 B_2(\tau_1)^2 - \sigma_3 B_2(\tau_1)^2\right)}{(2\zeta_1 - \sigma_2)(2\zeta_1 + \sigma_2)} - \frac{e^{-2i\zeta_1\tau_0}\left(\zeta_1{}^2\sigma_6\tilde{B_2}(\tau_1)^2 - \sigma_3\tilde{B_2}(\tau_1)^2\right)}{(2\zeta_1 - \sigma_2)(2\zeta_2 + \sigma_2)} + \frac{i\zeta_1\tilde{\sigma_4} B_2(\tau_1) e^{i\zeta_1\tau_0}}{(\zeta_1 - \sigma_2)(\zeta_1 + \sigma_2)} \frac{i\zeta_1\tilde{\sigma_4} e^{-i\zeta_1\tau_0}\tilde{B_2}(\tau_1)}{(\zeta_1 - \sigma_2)(\zeta_1 + \sigma_2)} - \frac{\zeta_1\tilde{\sigma_7} B_1(\tau_1) B_2(\tau_1)^2 e^{i\tau_0(2\zeta_1+\sigma_2)}}{4(\zeta_1 + \sigma_2)} - \frac{\zeta_1\tilde{\sigma_7}\tilde{B_1}(\tau_1)\tilde{B2}(\tau_1)^2 e^{-i\tau_0(2\zeta_1+\sigma_2)}}{4(\zeta_1 + \sigma_2)} + \frac{i\sigma_5 B_2(\tau_1)\tilde{B_1}(\tau_1) e^{i\tau_0(\zeta_1-\sigma_2)}}{\zeta_1 - 2\sigma_2} - \frac{i\sigma_5 B_1(\tau_1)\tilde{B_2}(\tau_1) e^{-i\tau_0(\zeta_1-\sigma_2)}}{\zeta_1 - 2\sigma_2} - \frac{\zeta_1\tilde{\sigma_7} B_2(\tau_1)^2\tilde{B_1}(\tau_1) e^{i\tau_0(2\zeta_1-\sigma_2)}}{4(\zeta_1 - \sigma_2)} - \frac{\zeta_1\tilde{\sigma_7} B_1(\tau_1)\tilde{B_2}(\tau_1)^2 e^{-i\tau_0(2\zeta_1-\sigma_2)}}{4(\zeta_1 - \sigma_2)}, \quad \cdots (18)$$

$$\gamma_2 = -\frac{\tilde{\zeta_2} B_2(\tau_1)^3 e^{3i\zeta_1\tau_0}}{8\zeta_1{}^2} - \frac{\tilde{\zeta_2} e^{-3i\zeta_1\tau_0}\tilde{B_2}(\tau_1)^3}{8\zeta_1{}^2} + \frac{i\sigma_2\tilde{\zeta_3} B_1(\tau_1) e^{i\sigma_2\tau_0}}{(\sigma_2 - \zeta_1)(\zeta 1 + \sigma_2)} - \frac{i\sigma_2\tilde{\zeta_3} e^{-i\sigma_2\tau_0}\tilde{B_1}(\tau_1)}{(\sigma_2 - \zeta_1)(\zeta_1 + \sigma_2)} - \frac{e^{i\tau_0(\zeta_1+\sigma_2)}\left(\zeta_1{}^2\tilde{\zeta_5} B_1(\tau_1) B_2(\tau_1) + \zeta_1\zeta_4\sigma_2 B_1(\tau_1) B_2(\tau_1)\right)}{\sigma_2(2\zeta_1 + \sigma_2)} - \frac{e^{-i\tau_0(\zeta_1+\sigma_2)}\left(\zeta_1{}^2\tilde{\zeta_5}\tilde{B_1}(\tau_1)\tilde{B_2}(\tau_1) + \zeta_1\zeta_4\sigma_2\tilde{B_1}(\tau_1)\tilde{B_2}(\tau_1)\right)}{\sigma_2} (2\zeta_1 + \sigma_2) + \frac{e^{i\tau_0(\zeta_1-\sigma_2)}\left(\zeta_1^2\tilde{\zeta_5} B_2(\tau_1)\tilde{B_1}(\tau_1) - \zeta_1\zeta_4\sigma_2 B_2(\tau_1)\tilde{B_1}(\tau_1)\right)}{\sigma_2(2\zeta_1 - \sigma_2)} + \frac{e^{-i\tau_0(\zeta_1-\sigma_2)}\left(\zeta 1^2\tilde{\zeta_5} B_1(\tau_1)\tilde{B_2}(\tau_1) - \zeta_1\zeta_4\sigma_2 B1(\tau_1)\tilde{B_2}(\tau_1)\right)}{\sigma_2(2\zeta_1 - \sigma_2)} \quad \cdots (19)$$

Modulation Equations

The modulation equations are a group of four first-order ODEs that describe the modulation of amplitudes and phases, since the procedures for solving them are complemented by initial conditions.

These secular terms in α2 and γ2 follow:

$$\alpha_{2,s} = -2\zeta_1^2 B_1(\tau_1) B_2(\tau_1) \tilde{\sigma_7}(\tau_1) \tilde{B_2}(\tau_1) - 2i\sigma_2 \frac{\partial B_1(\tau_1)}{\partial \tau_1} \quad \cdots (20)$$

$$\gamma_{2,s} = 3\tilde{\zeta_2} B_2(\tau_1)^2 \tilde{B_2}(\tau_1) - 2i\zeta_1 \frac{\partial B_2(\tau_1)}{\partial \tau_1} \quad \cdots (21)$$

$$B_k = \frac{1}{2} a_k(\tau) e^{i\psi_k}, \quad \widetilde{B}_k = \frac{1}{2} a_k(\tau) e^{-i\psi_k}, \quad k = 1, 2\,. \quad \cdots (22)$$

$$\dot{a}_1(\tau) = 0, \quad \dot{a}_2(\tau) = 0, \quad \dot{\psi}_1(\tau) = \frac{\zeta_1^2 a_2(\tau)^2 \sigma_7}{4\sigma_2} \quad \cdots (23)$$

$$\dot{\psi}_2(\tau) = -\frac{3a_2(\tau)^2 \zeta_2}{8\zeta_1}\,.$$

Final asymptotic solution up to the second order approximations

Once we reconstituted the modulation equations for the nonresonant cases and took into account the established equations, we obtained the final asymptotic solution up to the second-order approximations

$$\alpha = \frac{a_2(\tau)^2\left(\zeta 1^2\sigma_6+\sigma_3\right)-2\sigma_1}{2\sigma_2^2}+a_1(\tau)\cos(\sigma_2\tau+\psi_1(\tau)) + \frac{a_2(\tau)^2\left(\sigma_3-\zeta_1^2\sigma_6\right)\cos(2(\zeta_1\tau+\psi_2(\tau)))}{8\zeta_1^2-2\sigma_2^2} - \frac{\zeta_1\sigma_7 a_1(\tau)a_2(\tau)^2\cos(\tau(2\zeta_1-\sigma 2)-\psi_1(\tau)+2\psi 2(\tau))}{16(\zeta_1-\sigma_2)} - \frac{\zeta_1\sigma_7 a_1(\tau)a_2(\tau)^2\cos(\tau(2\zeta_1+\sigma_2)+\psi_1(\tau)+2\psi_2(\tau))}{16(\zeta_1+\sigma_2)} - \frac{\zeta_1\sigma_4 a_2(\tau)\sin(\zeta_1\tau+\psi_2(\tau))}{\zeta_1^2-\sigma_2^2} - \frac{\sigma_5 a_1(\tau)a_2(\tau)\sin(\tau(\zeta_1-\sigma_2)-\psi_1(\tau)+\psi_2(\tau))}{2(\zeta_1-2\sigma_2)} - \frac{\sigma_5 a_1(\tau)a_2(\tau)\sin(\tau(\zeta_1+\sigma_2)+\psi_1(\tau)+\psi_2(\tau))}{2(\zeta_1+2\sigma_2)} \quad \cdots(24)$$

$$\gamma = a_2(\tau)\cos(\zeta_1\tau+\psi_2(\tau)) - \frac{\zeta_2 a_2(\tau)^3\cos(3(\zeta_1\tau+\psi_2(\tau)))}{32\zeta_1^2} + \frac{\zeta_1 a_1(\tau)a_2(\tau)(\zeta_1\zeta_5-\zeta_4\sigma_2)\cos(\tau(\zeta_1-\sigma_2)-\psi_1(\tau)+\psi_2(\tau))}{2\sigma_2(2\zeta_1-\sigma_2)} - \frac{\zeta_1 a_1(\tau)a_2(\tau)(\zeta_1\zeta_5+\zeta_4\sigma_2)\cos(\tau(\zeta_1+\sigma_2)+\psi 1(\tau)+\psi_2(\tau))}{2\sigma_2(2\zeta_1+\sigma 2)} + \frac{\zeta_3\sigma_2 a_1(\tau)\sin(\sigma_2\tau+\psi_1(\tau))}{\zeta_1^2-\sigma_2^2}. \quad \cdots(25)$$

Comparison between analytical and numerical solution using time histories

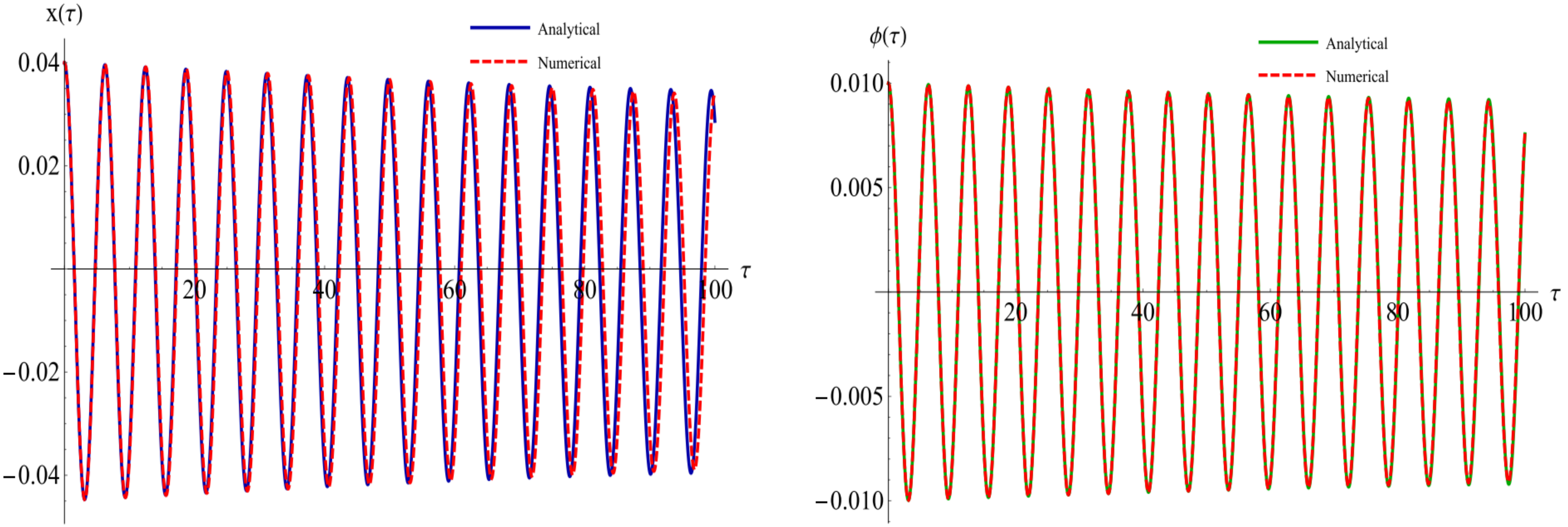

Fig.2 *Comparison between the analytical and numerical solution of the SAM*

$\sigma_1 = 0.0025, \ \sigma_2 = 1.01, \ \sigma_3 = 0.06, \ \sigma_4 = 0.014, \ \zeta_2 = 0.2, \ \zeta_3 = 0.0005, \ \zeta_4 = 0.01, \ \zeta_5 = 0.00001,$

$\sigma_5 = 0.06, \ \sigma_6 = 0.01, \ \sigma_7 = 0.05, \ \zeta_1 = 1.0, \qquad x(\tau) = 0.04, \ \dot{x}(\tau)0, \ \phi(\tau) = 0.01, \ \dot{\phi}(\tau) = 0$

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The modified SAM physical model and the governing equations

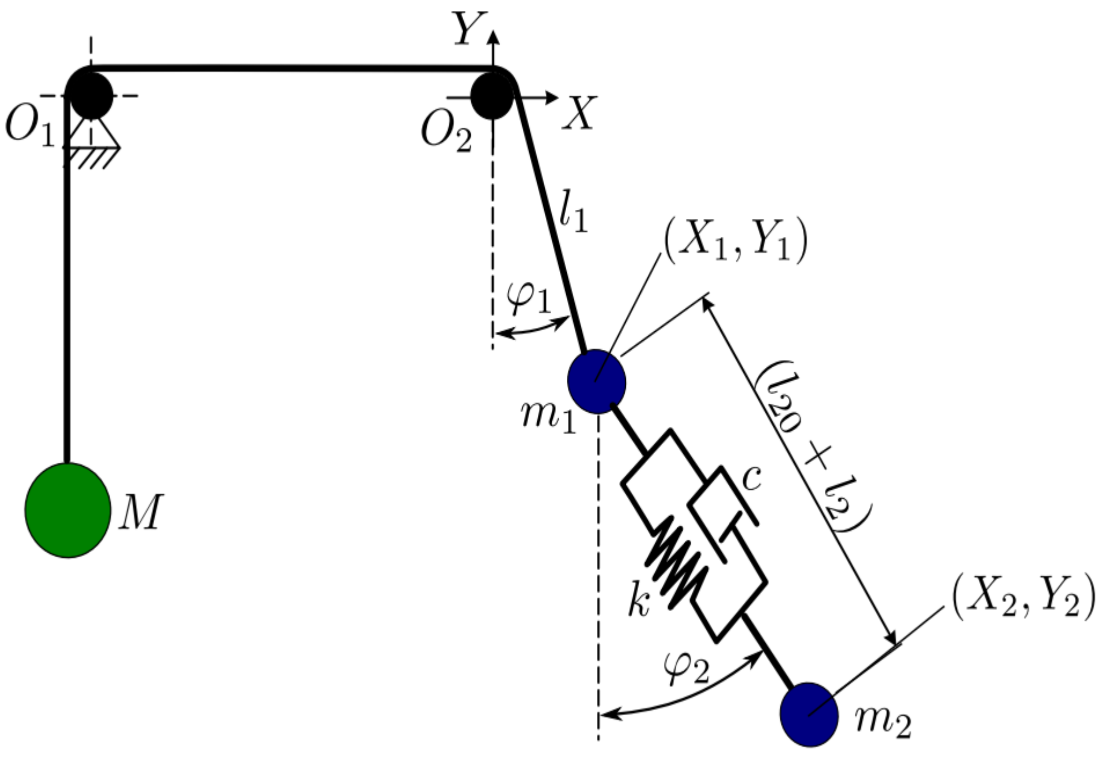

Fig.3 *The extended swinging Atwood machine*

$$\ddot{l}_1(t) = \frac{-(m_1 s_{f2} + M)\ddot{X}_0 + m_1\left(l_1(t)\dot{\varphi}_1^2(t) + g c_{f1}\right) + T_{t,3}\left(c_{f1}c_{f2} - s_{f1}s_{f2}\right) + T_R - Mg}{m_1 + M},$$

$$\ddot{l}_2(t) = \frac{1}{m_1 m_2 (m_1 + M)}\Big(-((s_{f2}(s_{f1}^2 - s_{f1}) + c_{f1}c_{f2}s_{f1} - c_{f1}c_{f2})M m_1 m_2 \ddot{X}_0 +$$
$$M m_2 s_{f1}^2 T_{t,3}(1 - 2s_{f1}^2) + M m_2(m_1 l_1(t)\dot{\varphi}_1^2(t) + 2c_{f1}c_{f2}T_{t,3} + g m_1 c_{f1}) +$$
$$m_1 m_2 s_{f1} s_{f2}(T_R - Mg) + M m_2 s_{f1}^2(T_{t,3} + g m_1 c_{f2}) - (m_1 m_2(m_1 + M)(l_2(t) +$$
$$l_{20s}))\dot{\varphi}_2^2(t) - M m_1 m_2 c_{f1} c_{f2} l_1(t)\dot{\varphi}_1^2(t) + (m_2 + m_1 + M)m_1 T_{t,3} + m_1 m_2 c_{f2}(c_{f1}T_R$$
$$- M g c_{f1} - Mg))\Big),$$

$$\ddot{\varphi}_1(t) = -\frac{m_1\left(2\dot{l}_1(t)\dot{\varphi}_1(t) + c_{f1}\ddot{X}_0 + g s_{f1}\right) + T_{t,3}\left(s_{f1}c_{f2} - s_{f2}c_{f1}\right)}{m_1 l_1(t)},$$

$$\ddot{\varphi}_2(t) = \frac{1}{(m_1 + M)(m_1 l_2(t) + m_1 l_{20s})}\Big(((s_{f1} - 1)(s_{f2}c_{f1} - s_{f1}c_{f2}))M m_1 \ddot{X}_0 -$$
$$2M s_{f1} c_{f1} s_{f2}^2 T_{t,3} + ((2c_{f2}s_{f1}^2 T_{t,3} + g m_1 s_{f1}^2 - m_1 c_{f1} l_1(t)\dot{\varphi}_1^2(t) - c_{f2}T_{t,3})M +$$
$$(T_R - Mg)m_1 c_{f1} - M g m_1)s_{f2} + ((m_1 c_{f2} l_1(t)\dot{\varphi}_1^2(t) + c_{f1}T_{t,3})M + (M g c_{f1} -$$
$$T_R + Mg)m_1 c_{f2})s_{f1} - 2m_1 \dot{l}_2(t)\dot{\varphi}_2^2(t)(m_1 + M)\Big), \quad \cdots (26)$$

Assumptions

$$\sigma_1 = \omega^2\lambda + G\omega_4^2 - \frac{\omega^2\omega_1^2\omega_4^2}{\lambda^2},\ \sigma_2 = FG,\ \sigma_3 = \frac{F\omega^2\omega_1^2}{\lambda^2},$$

$$\sigma_4 = \frac{F\omega^2\omega_1^2}{6\lambda^2},\ \sigma_5 = \frac{\omega^2\omega_1^2\omega_4^2}{2\lambda^2},\ \sigma_6 = \omega^2\omega_1,\ \sigma_7 = \frac{\omega^2\omega_1^2}{\lambda},\ \sigma_8 = \frac{2\omega^2\omega_1^2}{\lambda},$$

$$\sigma_9 = \frac{\omega^2\omega_1^2}{\lambda^2}, \delta_1 = \frac{G\lambda^3}{\omega_1^2} + \frac{5G\omega_4^2}{2},\ \delta_2 = \frac{G\lambda^2}{\omega_1^2},\ \delta_2 = FG\omega_0,\ \delta_4 = \frac{G\omega_4^2}{4},$$

$$\delta_5 = \frac{FG}{12},\ \delta_6 = \frac{2G\lambda}{\omega_1}, \delta_7 = b_2c_1,\ \delta_8 = b_3c_1,\ \delta_9 = \frac{2G\lambda^2}{\omega_1},\ \delta_0 = G\lambda$$

$$\zeta_1 = \frac{F}{3} + \frac{5G\omega_4^2}{2},\ \zeta_2 = \frac{\omega_4^2}{6},\ \zeta_3 = \frac{2}{\omega_1},\ \zeta_4 = \frac{1}{\lambda}\xi_1 = \frac{G\lambda^3}{A\omega_1^2} + 2G\omega_4^2,$$

$$\xi_2 = \frac{G\lambda^2}{A\omega_1^2},\ \xi_3 = G\omega_5^2,\ \xi_4^2 = \frac{G\lambda^3}{A\omega_1^2} + G\omega_4^2, \xi_5 = \frac{hFG}{4},\ \xi_6 = \frac{G\omega_4^2}{12},$$

$$\xi_7 = c_2G, h = 1,\ \xi_8 = \frac{2G\lambda^2}{A\omega_1},\ \xi_9 = \frac{2G\lambda}{A\omega_1},\ \xi_{10} = \frac{G\lambda}{A},\ \xi_{11} = \frac{G}{A},$$

$$\xi_{12} = \frac{2G}{A},\ \xi_{13} = \frac{2G_1}{A},\ \xi_{14} = \frac{1}{A}.$$

$$\begin{aligned}
&\sin\phi_i(t) = \left(\phi_i(t) - \frac{(\phi_i(t))^3}{6}\right),\\
&\cos\phi_i = \left(1 - \frac{(\phi_i(t))^2}{2}\right),\\
&\sin(2(\phi_{i+1}(t) - \phi_i(t))) = 2(\phi_{i+1}(t) - \phi_i(t)),\\
&\cos(2(\phi_{i+1}(t) - \phi_i(t))) = 1\\
&\sin(\phi_{i+1}(t) - \phi_i(t)) = (\phi_{i+1}(t) - \phi_i(t)),\\
&\cos(\phi_{i+1}(t) - \phi_i(t)) = 1,\\
&\sin(\phi_{i+1}(t) - 2\phi_i(t)) = (\phi_{i+1}(t) - 2\phi_i(t)),\\
&\cos(\phi_{i+1}(t) - 2\phi_i(t)) = 1.
\end{aligned} \quad \cdots (27)$$

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Analytical Solution

The final dimensionless form of the motion equations

$$\sigma_1 - \sigma_2 \sin(\bar{\omega}\tau) - w^2 x_1(\tau) - \omega_0 x_2(\tau) - \sigma_3 \sin(\bar{\omega}\tau)\,\phi_1(\tau) + \sigma_4 \sin(\bar{\omega}\tau)\,\phi_1^3(\tau) + \sigma_5 \phi_2^2(\tau) - c_1\dot{x}_2(\tau) - \sigma_6\phi_1(\tau) - \sigma_7\dot{\phi}_1^2(\tau) - \sigma_8 x_1(\tau)\dot{\phi}_1(\tau) - \sigma_9 x_1(\tau)\dot{\phi}_1^2(\tau) - \ddot{x}_1(\tau) = 0$$

$$\delta_1 - \sigma_2 \sin(\bar{\omega}\tau) + \delta_2 x_1(\tau) - x_2(\tau) - b_2 x_2(\tau) - b_2 x_2(\tau) - b_3 x_2(\tau) + \delta_3 \sin(\bar{\omega}\tau)\,\phi_1(\tau) - \delta_4\phi_2^2(\tau) - \delta_5 \sin(\bar{\omega}\tau)\phi_2^3(\tau) - c_1\dot{x}_2(\tau) - \delta_7\dot{x}_2(\tau) - \delta_8\dot{x}_2(\tau) + \delta_9\dot{\phi}_1(\tau) + \delta_6 x_1(\tau)\dot{\phi}_1(\tau) + \delta_0\dot{\phi}_1^2(\tau) + G x_1(\tau)\dot{\phi}_1^2(\tau) + \frac{1}{2}AG(\tau)\dot{\phi}_2^2(\tau) + AG_1(\tau)\dot{\phi}_2^2(\tau) + G_1 x_2(\tau)\dot{\phi}_2^2(\tau) - \ddot{x}_2(\tau) = 0 \quad \cdots (28)$$

$$F\sin(\bar{\omega}\tau) - \omega_4^2\phi_1 + \omega_5^2 x_2(\tau)\phi_1(\tau) - \zeta_1 \sin(\bar{\omega}\tau)\,\phi_1^2(\tau) + \zeta_1\phi_1^3(\tau) - \omega_5^2 x_2(\tau)\phi_2(\tau) - \zeta_3\dot{x}_1(\tau) - c_2\phi_1(\tau)\dot{x}_2(\tau) + c_2\phi_2(\tau)\dot{x}_2(\tau) - 2\zeta_4\dot{x}_1(\tau)\dot{\phi}_1(\tau) - \zeta_4 x_1(\tau)\ddot{\phi}_1(\tau) - \ddot{\phi}_2(\tau) = 0$$

$$\xi_1\phi_1(\tau) + h\sigma_2 \sin(\bar{\omega}\tau)\,\phi_1(\tau) + \xi_2 x_1(\tau)\phi_1(\tau) + \xi_3 x_1(\tau)\phi_1(\tau) - \xi_4^2\phi_2(\tau) - h\sigma_2 \sin(\bar{\omega}\tau)\,\phi_2(\tau) - \xi_2 x_1(\tau)\phi_2(\tau) - \xi_3 x_2(\tau)\phi_2(\tau) - \xi_5 \sin(\bar{\omega}\tau)\,\phi_2^2(\tau) - \xi_6\phi_2^3(\tau) + \xi_7\phi_1(\tau)\dot{x}_2(\tau) - \xi_7\phi_2(\tau)\dot{x}_2(\tau) + \xi_8\phi_1(\tau)\dot{\phi}_2^2(\tau) + \xi_9 x_1(\tau)\phi_1(\tau)\dot{\phi}_2^2(\tau) - \xi_8\phi_2(\tau)\dot{\phi}_1^2(\tau) - \xi_9 x_1(\tau)\phi_2(\tau)\dot{\phi}_1^2(\tau) + \xi_{10}\phi_1(\tau)\dot{\phi}_1^2(\tau) + \xi_{11} x_1(\tau)\phi_1(\tau)\dot{\phi}_1^2(\tau) - \xi_{10}\phi_2(\tau)\dot{\phi}_1^2(\tau) - \xi_{11} x_1(\tau)\phi_2(\tau)\dot{\phi}_1^2(\tau) + \xi_{12}\dot{x}_2(\tau)\dot{\phi}_2^2(\tau) + \xi_{13}\dot{x}_2(\tau)\dot{\phi}_2^2(\tau) - \xi_{14} x_2(\tau)\ddot{\phi}_2(\tau) - \ddot{\phi}_2(\tau) = 0$$

Multiple scale approach technique

The analysis focused on a localized region near the system's static equilibrium. A small parameter represented as $0 < \varepsilon \ll 1$ is introduced to characterize the amplitudes of the oscillations within this region.

This parameter allows us to establish the following relationship:

$$x_1(\tau) = \varepsilon\alpha(\tau:\varepsilon),\ \ x_2(\tau) = \varepsilon\beta(\tau:\varepsilon),$$

$$\phi_1(\tau) = \varepsilon\gamma(\tau:\varepsilon),\ \ \phi_2(\tau) = \varepsilon\Gamma(\tau:\varepsilon). \qquad \cdots(29)$$

This allows us to consider the following approximation:

$$x_1(\tau) = \varepsilon\alpha(\tau_0,\ \tau_1),\ x_2(\tau) = \varepsilon\beta(\tau_0,\ \tau_1),\ \phi_1(\tau) = \varepsilon\gamma(\tau_0,\ \tau_1),$$

$$\phi_2(\tau) = \varepsilon\Gamma(\tau_0,\ \tau_1)\,\dot{x}_1(\tau) = \varepsilon\dot{\alpha}(\tau_0,\ \tau_1),\ \dot{x}_2(\tau) = \varepsilon\dot{\beta}(\tau_0,\ \tau_1),$$

$$\dot{\phi}_1(\tau) = \varepsilon\dot{\gamma}(\tau_0,\ \tau_1),\ \dot{\phi}_2(\tau) = \varepsilon\dot{\Gamma}(\tau_0,\ \tau_1),\ \ddot{x}_1(\tau) = \varepsilon\ddot{\alpha}(\tau_0,\ \tau_1),$$

continued on next page

Multiple scale approach technique…

$$\ddot{x}_2(\tau) = \varepsilon\ddot{\beta}(\tau_0, \tau_1), \ddot{\phi}_1(\tau) = \varepsilon\ddot{\gamma}(\tau_0, \tau_1), \ddot{\phi}_2(\tau) = \varepsilon\ddot{\Gamma}(\tau_0, \tau_1)\, b = \varepsilon\tilde{b},$$

$$F = \varepsilon\tilde{F}, c_1 = \varepsilon\tilde{c}_1, G_2 = \varepsilon\tilde{G}_2, \omega_0 = \varepsilon\tilde{\omega}_0, \sigma_1 = \varepsilon^2\tilde{\sigma}_1, \sigma_2 = \varepsilon^2\tilde{\sigma}_2,$$

$$\sigma_3 = \varepsilon\tilde{\sigma}_3, \sigma_4 = \varepsilon^{-1}\tilde{\sigma}_4, \sigma_6 = \varepsilon\tilde{\sigma}_6, \sigma_9 = \varepsilon^{-1}\tilde{\sigma}_9, A = \varepsilon\tilde{A}, b_2 = \varepsilon\tilde{b}_2,$$

$$b_3 = \varepsilon\tilde{b}_3, G = \varepsilon^{-1}\tilde{G}, G_1 = \varepsilon^{-1}\tilde{G}_1, \delta_2 = \varepsilon\tilde{\delta}_2, \delta_3 = \varepsilon\tilde{\delta}_3, \delta_5 = \varepsilon^{-1}\tilde{\delta}_5, \quad \cdots (30)$$

$$\delta_7 = \varepsilon\tilde{\delta}_7, \delta_8 = \varepsilon\tilde{\delta}_8, \delta_9 = \varepsilon\tilde{\delta}_9, \delta_1 = \varepsilon^2\tilde{\delta}_1, y = \varepsilon\tilde{y}, c_2 = \varepsilon\tilde{c}_2, \zeta_3 = \varepsilon\tilde{\zeta}_3,$$

$$\zeta_2 = \varepsilon^{-1}\tilde{\zeta}_2, h = \varepsilon^{-1}h, \xi_1 = \varepsilon\tilde{\xi}_1, \xi_6 = \varepsilon^{-1}\tilde{\xi}_6, \xi_9 = \varepsilon^{-1}\tilde{\xi}_9, \xi_{10} = \varepsilon^{-1}\tilde{\xi}_{10},$$

$$\xi_{11} = \varepsilon^{-2}\tilde{\xi}_{11}.$$

Multiple scale approach technique…

The time-dependent variable $x_1(\tau)$, $x_2(\tau)$, $\phi_1(\tau)$, and $\phi_2(\tau)$ can be considered as a power series of ε

$$x_1(\tau) = \sum_{k=1}^{2} \varepsilon^k x_{1,k}(\tau_0, \tau_1) + O(\varepsilon^k),$$

$$x_2(\tau) = \sum_{k=1}^{2} \varepsilon^k x_{2,k}(\tau_0, \tau_1) + O(\varepsilon^k),$$

$$\phi_1(\tau) = \sum_{k=1}^{2} \varepsilon^k \phi_{1,k}(\tau_0, \tau_1) + O(\varepsilon^k),$$

$$\phi_2(\tau) = \sum_{k=1}^{2} \varepsilon^k \phi_{2,k}(\tau_0, \tau_1) + O(\varepsilon^k). \quad \cdots (31)$$

The time scales are represented by $\tau_n = \varepsilon^n \tau \ (n = 0, 1)$, τ_0 – faster; τ_1 – slowest

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Multiple scale approach technique…

The following operators are used to convert derivatives with respect to τ to the new time scales τ_n.

$$\frac{d}{d\,\tau} = \frac{\partial}{\partial \tau_0} + \varepsilon \frac{\partial}{\partial \tau_0}, \quad \frac{d^2}{d\tau^2} = \frac{\partial^2}{\partial \tau_0^2} + 2\varepsilon \frac{\partial^2}{\partial \tau_0 \partial \tau_1} + O\left(\varepsilon^2\right) \qquad \cdots (32)$$

(ii) First-order equations (coefficient “1” at ε^1)

$$\frac{\partial^2 \alpha_1}{\partial \tau_0^2} + w^2 \alpha_1 = 0, \qquad \cdots (33)$$

$$\frac{\partial^2 \beta_1}{\partial \tau_0^2} + \beta_1 = 0, \qquad \cdots (34)$$

$$\frac{\partial^2 \gamma_1}{\partial \tau_0^2} + \omega_4^2 \gamma_1 = 0, \qquad \cdots (35)$$

$$\frac{\partial^2 \Gamma_1}{\partial \tau_0^2} + \zeta_4^2 \Gamma_1 = 0. \qquad \cdots (36)$$

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(ii) Second-order equations (coefficient "2" at ε^2)

$$\widetilde{\sigma}_1 - \widetilde{\sigma}_2 \sin(\omega\tau_0) - w^2\alpha_1 - \omega_0\beta_1 - \widetilde{\sigma}_3\gamma_1 \sin(\omega\tau_0) + \widetilde{\sigma}_4\gamma_1^3 \sin(\omega\tau_0) + \widetilde{\sigma}_5\Gamma_1^2 \sin(\omega\tau_0) - \widetilde{c}_1\frac{\partial\beta_1}{\partial\tau_0} - \widetilde{\sigma}_6\frac{\partial\gamma_1}{\partial\tau_0} - \sigma_8\alpha_1\frac{\partial\gamma_1}{\partial\tau_0} - \sigma_7\left(\frac{\partial\gamma_1}{\partial\tau_0}\right)^2 - \widetilde{\sigma}_6\alpha_1\left(\frac{\partial\gamma_1}{\partial\tau_0}\right)^2 - 2\frac{\partial^2\alpha_1}{\partial\tau_0\partial\tau_1} - \frac{\partial^2\alpha_2}{\partial\tau_0^2} = 0, \quad \cdots (37)$$

$$\widetilde{\delta}_1 - \widetilde{\delta}_2 \sin(\omega\tau_0) + \widetilde{\delta}_2\alpha_1 - \widetilde{b}_2\beta_1 - \widetilde{b}_3\beta_1 - \beta_1 + \widetilde{\delta}_3\gamma_1 \sin(\omega\tau_0) - \widetilde{\delta}_4\Gamma_1^2 - \widetilde{\delta}_5\Gamma_1^3 \sin(\omega\tau_0) - \widetilde{c}_1\frac{\partial\beta_1}{\partial\tau_0} - \widetilde{\delta}_7\frac{\partial\beta_1}{\partial\tau_0} - \widetilde{\delta}_8\frac{\partial\beta_1}{\partial\tau_0} + \widetilde{\delta}_9\frac{\partial\gamma_1}{\partial\tau_0} + \delta_6\alpha_1\frac{\partial\gamma_1}{\partial\tau_0} + \delta_0\left(\frac{\partial\gamma_1}{\partial\tau_0}\right)^2 + \widetilde{G}\alpha_1\left(\frac{\partial\gamma_1}{\partial\tau_0}\right)^2 + \frac{1}{2}\widetilde{A}\widetilde{G}\left(\frac{\partial\Gamma_1}{\partial\tau_0}\right)^2 + \widetilde{A}\widetilde{G}_1\left(\frac{\partial\Gamma_1}{\partial\tau_0}\right)^2 + \widetilde{G}\beta_1\left(\frac{\partial\Gamma_1}{\partial\tau_0}\right)^2 + \widetilde{G}_1\beta_1\left(\frac{\partial\Gamma_1}{\partial\tau_0}\right)^2 - 2\frac{\partial^2\beta_1}{\partial\tau_0\partial\tau_1} - \frac{\partial^2\beta_2}{\partial\tau_0^2} = 0, \quad \cdots (38)$$

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(ii) Second-order equations (coefficient "2" at ε^2) ...

$$\widetilde{F}\sin(\varpi\tau_0)+\omega_5^2\beta_1\gamma_1-\zeta_1\gamma_1^2\sin(\varpi\tau_0)+\zeta_2\gamma_1^3-\omega_4^2\gamma_2-\omega_5^2\beta_1\Gamma_1-$$

$$\zeta_3\frac{\partial\alpha_1}{\partial\tau_0}-2\zeta_4\frac{\partial\alpha_1}{\partial\tau_0}\frac{\partial\gamma_1}{\partial\tau_0}-2\frac{\partial^2\gamma_1}{\partial\tau_0\partial\tau_1}-\zeta_4\alpha_1\frac{\partial^2\gamma_1}{\partial\tau_0^2}-\frac{\partial^2\gamma_2}{\partial\tau_0^2}=0, \quad \cdots (39)$$

$$\widetilde{\xi}_1\gamma_1+\widetilde{h}\widetilde{\sigma}_2\gamma_1\sin(\varpi\tau_0)+\xi_2\alpha_1\gamma_1+\xi_2\beta_1\gamma_1-\widetilde{h}\widetilde{\sigma}_2\Gamma_1\sin(\varpi\tau_0)-$$

$$\xi_2\alpha_1\Gamma_1-\xi_3\beta_1\Gamma_1-\xi_5\Gamma_1^2\sin(\varpi\tau_0)-\widetilde{\xi}_6\Gamma_1^3-\xi_4^2\Gamma_2+\xi_7\gamma_1\frac{\partial\beta_1}{\partial\tau_0}-$$

$$\xi_7\Gamma_1\frac{\partial\beta_1}{\partial\tau_0}+\xi_8\gamma_1\frac{\partial\gamma_1}{\partial\tau_0}+\widetilde{\xi}_9\alpha_1\gamma_1\frac{\partial\gamma_1}{\partial\tau_0}-\xi_8\Gamma_1\frac{\partial\gamma_1}{\partial\tau_0}-\widetilde{\xi}_9\alpha_1\Gamma_1\frac{\partial\gamma_1}{\partial\tau_0}+ \quad \cdots (40)$$

$$\widetilde{\xi}_{10}\gamma_1\left(\frac{\partial\gamma_1}{\partial\tau_0}\right)^2+\widetilde{\xi}_{11}\alpha_1\gamma_1\left(\frac{\partial\gamma_1}{\partial\tau_0}\right)^2-\widetilde{\xi}_{10}\Gamma_1\left(\frac{\partial\gamma_1}{\partial\tau_0}\right)^2-\widetilde{\xi}_{11}\alpha_1\Gamma_1\left(\frac{\partial\gamma_1}{\partial\tau_0}\right)^2+$$

$$\xi_{12}\frac{\partial\beta_1}{\partial\tau_0}\frac{\partial\Gamma_1}{\partial\tau_0}+\xi_{13}\frac{\partial\beta_1}{\partial\tau_0}\frac{\partial\Gamma_1}{\partial\tau_0}-2\frac{\partial\Gamma_1}{\partial\tau_0\partial\tau_1}-\xi_{14}\beta_1\frac{\partial^2\Gamma_1}{\partial\tau_0^2}-\frac{\partial^2\Gamma_2}{\partial\tau_0^2}=0,$$

Multiple scale approach technique

- The solutions to the equations above (37) - (40), which can be solved in a specific order, emphasize the importance of the solutions from the first group. Therefore, we first focus on obtaining the general solutions of Eqs. (33) - (36). The established solutions are as follows:

$$\alpha_1 = e^{iw\tau_0} B_1(\tau_1) + e^{-iw\tau_0}\widetilde{B}_1, \qquad \gamma_1 = e^{i\omega_4\tau_0} B_3 + e^{-i\omega_4\tau_0}\widetilde{B}_3,$$

$$\beta_1 = e^{i\tau_0} B_2 + e^{-i\tau_0}\widetilde{B}_2, \qquad \Gamma_1 = e^{i\xi_4\tau_0} B_4 + e^{-i\xi_4\tau_0}\widetilde{B}_4. \quad \cdots (41)$$

- Consequently, by substituting the solutions (41) into the second group of PDEs (37)–(40), we obtain the second-order solutions with B_i and $\tilde{B}_i$being τ_1 dependant where i = 1, 2, 3, 4:

Modulation equations

The modulation equations are a group of four first-order ODEs that describe the modulation of amplitudes and phases, since the procedures for solving them are complemented by initial conditions. These secular terms in $\boldsymbol{\alpha_2}$, $\boldsymbol{\beta_2}$, $\boldsymbol{\gamma_2}$, and $\boldsymbol{\Gamma_2}$ follow:

$$\alpha_{2,s} = -2\omega_4^2\tilde{\sigma_9}B_1(\tau_1)B_3(\tau_1)\tilde{B_3}(\tau_1) - 2iw\frac{\partial B_3(\tau_1)}{\partial\tau_1} \quad \cdots (42)$$

$$\beta_{2,s} = -\tilde{b_2}B_2(\tau_1) - \tilde{b_3}B_2(\tau_1) + 2\xi_4^2\tilde{G}B_2(\tau_1)B_4(\tau_1)\tilde{B_4}(\tau_1) + 2\xi_4{}^2\tilde{G_1}B_2(\tau_1)B_4(\tau_1)\tilde{B_4}(\tau_1) - i\tilde{c_1}B_2(\tau_1) - i\tilde{\delta_7}B_2(\tau_1) - i\tilde{\delta_8}B_2(\tau_1) - 2i\frac{\partial B_2(\tau_1)}{\partial\tau_1} \quad \cdots (43)$$

$$\gamma_{2,s} = 3\tilde{\zeta_2}B_3(\tau_1)^2\tilde{B_3}(\tau_1) - 2i\omega_4\frac{\partial B_3(\tau_1)}{\partial\tau_1} \quad \cdots (44)$$

$$\Gamma_{2,s} = -2\omega_4{}^2\tilde{\xi_1}0B_3(\tau_1)B_4(\tau_1)\tilde{B_3}(\tau_1) - 3\tilde{\xi_6}B_4(\tau_1)^2\tilde{B_4}(\tau_1) - 2i\xi_4\frac{\partial B_4(\tau_1)}{\partial\tau_1} \quad \cdots (45)$$

$$B_k = \frac{1}{2}a_k(\tau)e^{i\psi_k}, \widetilde{B}_k = \frac{1}{2}a_k(\tau)e^{-i\psi_k} \quad \text{for} \quad k = 1\ldots4.$$

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Modulation Equations…

In order $\boldsymbol{\Psi_j}$ and $\boldsymbol{a_j}$ represent the phases and amplitude of the solutions $\boldsymbol{\alpha, \beta, \gamma,}$ and $\mathbf{\Gamma}$.

After eliminating the secular terms from $\boldsymbol{\alpha_2, \beta_2, \gamma_2,}$ and $\boldsymbol{\Gamma_2}$, we obtained the following modulation equations

$$\dot{a}_1(\tau) = 0, \quad \dot{a}_2(\tau) = -\frac{1}{2}a_2(\tau)(c_1+\delta_7+\delta_8=0), \quad \dot{a}_3(\tau)=0, \quad \dot{a}_4(\tau)=0,$$

$$\dot{\psi}_1(\tau) = \frac{\omega_4^2 a_3(\tau)^2\sigma_9}{4w}, \quad \dot{\psi}_2(\tau) = \frac{1}{4}\left(2b_2+2b_3+\xi_4^2 a_4(\tau)^2(G+G_1)\right), \quad \cdots (46)$$

$$\dot{\psi}_3(\tau) = -\frac{3a_3(\tau)^2\zeta_2}{8\omega_4}, \quad \dot{\psi}_4(\tau) = \frac{2\omega_4^2 a_3(\tau)^2\xi_{10}+3a_4(\tau)^2\xi_6}{8\xi_4}.$$

After reconstitution of the modulation equations for the nonresonant cases and taking into account the established equations (41), the final asymptotic solution up to the second order approximations with $\boldsymbol{\Psi_i}$ and $\boldsymbol{a_i}$ being $\boldsymbol{\tau_1}$ dependent for **i** = 1, 2, 3, 4 is as $\boldsymbol{\alpha, \beta, \gamma,}$ and $\mathbf{\Gamma}$.

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Comparison between analytical and numerical solution using time history

$$A = 0.5, c_1 = c_2 = \xi_1 = \xi_3 = \delta_3 = 0.01,\ \sigma_5 = \sigma_8 = \zeta_3 = 0.0001,$$

$$\omega_5 = \xi_2 = 0.002,\ \sigma_6 = \sigma_7 = \xi_{10} = \xi_{11} = 0.0002,\ \xi_2 = 0.00021,$$

$$\sigma_9 = \xi_{14} = \zeta_4 = 0.00005,\ \delta_4 = 0.003,\ \delta_4 = 0.00008,\ \xi_7 = 0.0012, \qquad \cdots (47)$$

$$h = \omega_0 = 1,\ \delta_8 = 0.008,\ \delta_9 = 0.00009,\ \delta_0 = 0.00002,\ \xi_6 = 0.00008,$$

$$\xi_9 = 0.000002,\ \xi_{13} = 0.00015,\ \varpi = 10,\ \sigma_1 = 0.15,\ \sigma_2 = 0.464,\ b_2 = 2.11,$$

$$b_3 = 1.63,\ G = 0.8,\ G_1 = 0.1,\ F = 0.81,\ \omega_4 = 1.72,\ \zeta_1 = 0.05,\ \sigma_3 = 1.15,$$

$$w = 0.25,\ \sigma_2 = 0.001,\ \xi_4 = 1.61,\ \xi_5 = 0.005\,.$$

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Comparison between analytical and numerical solution using time history…

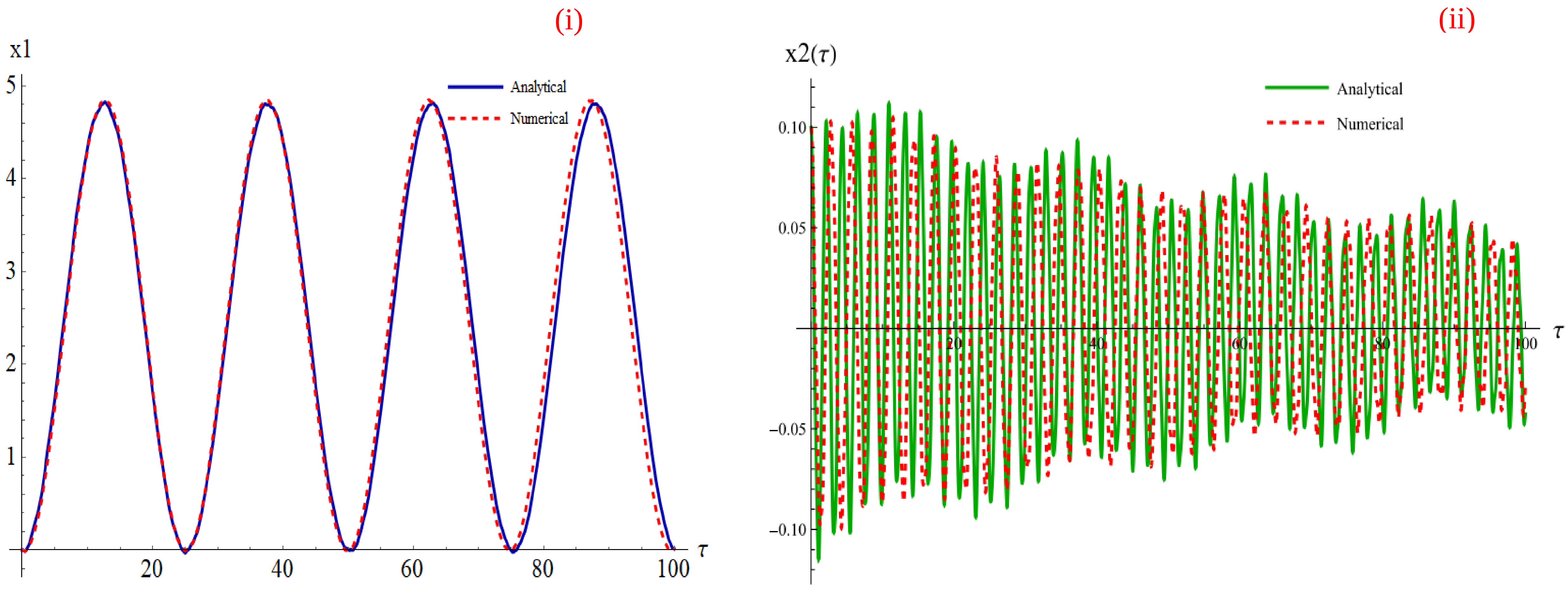

Fig.4a Comparison between the analytical and numerical solution of the 4-DOF modified SAM for $x_1(\tau)$ (i), and $x_2(\tau)$ (ii)

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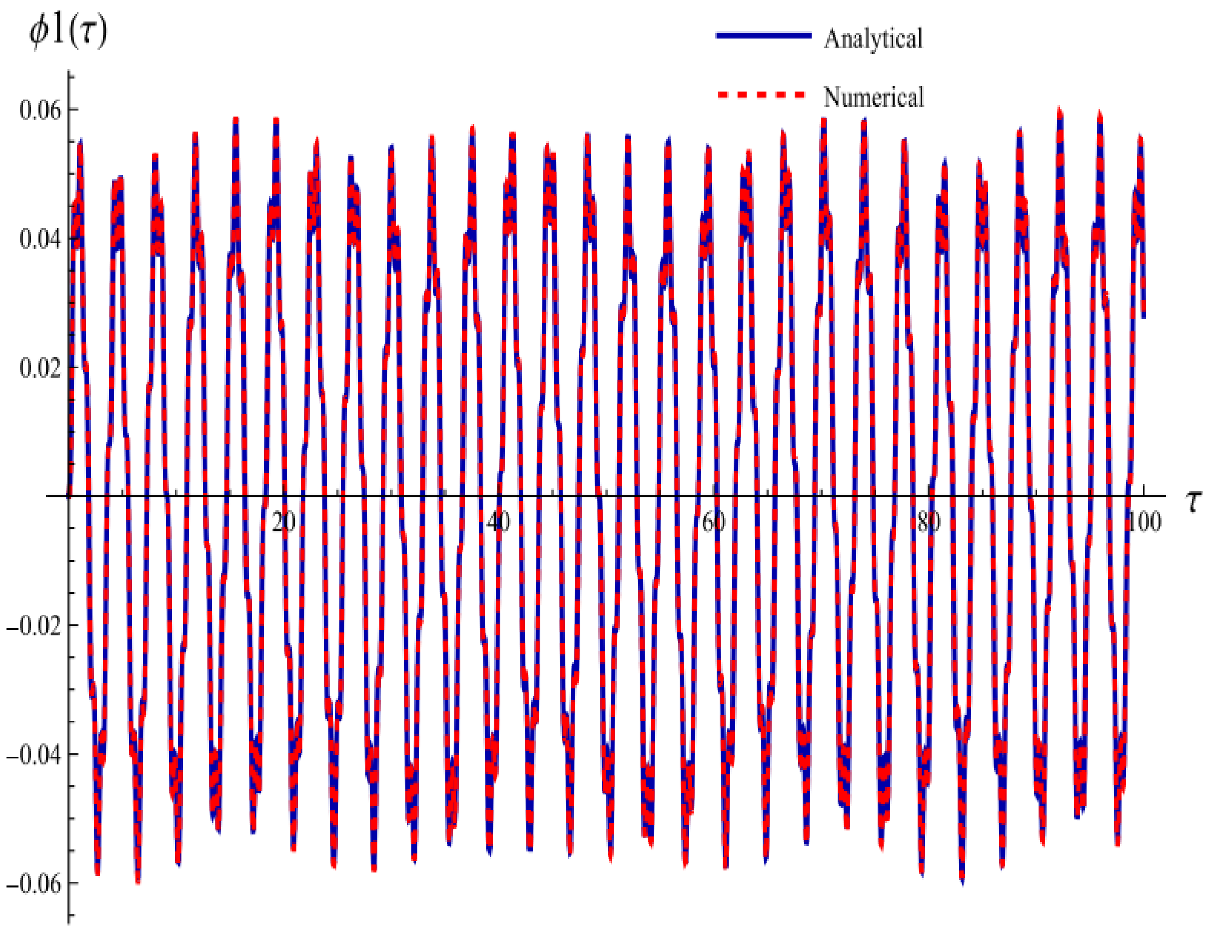
Comparison between analytical and numerical solution using time history…

(iii)



(iv)

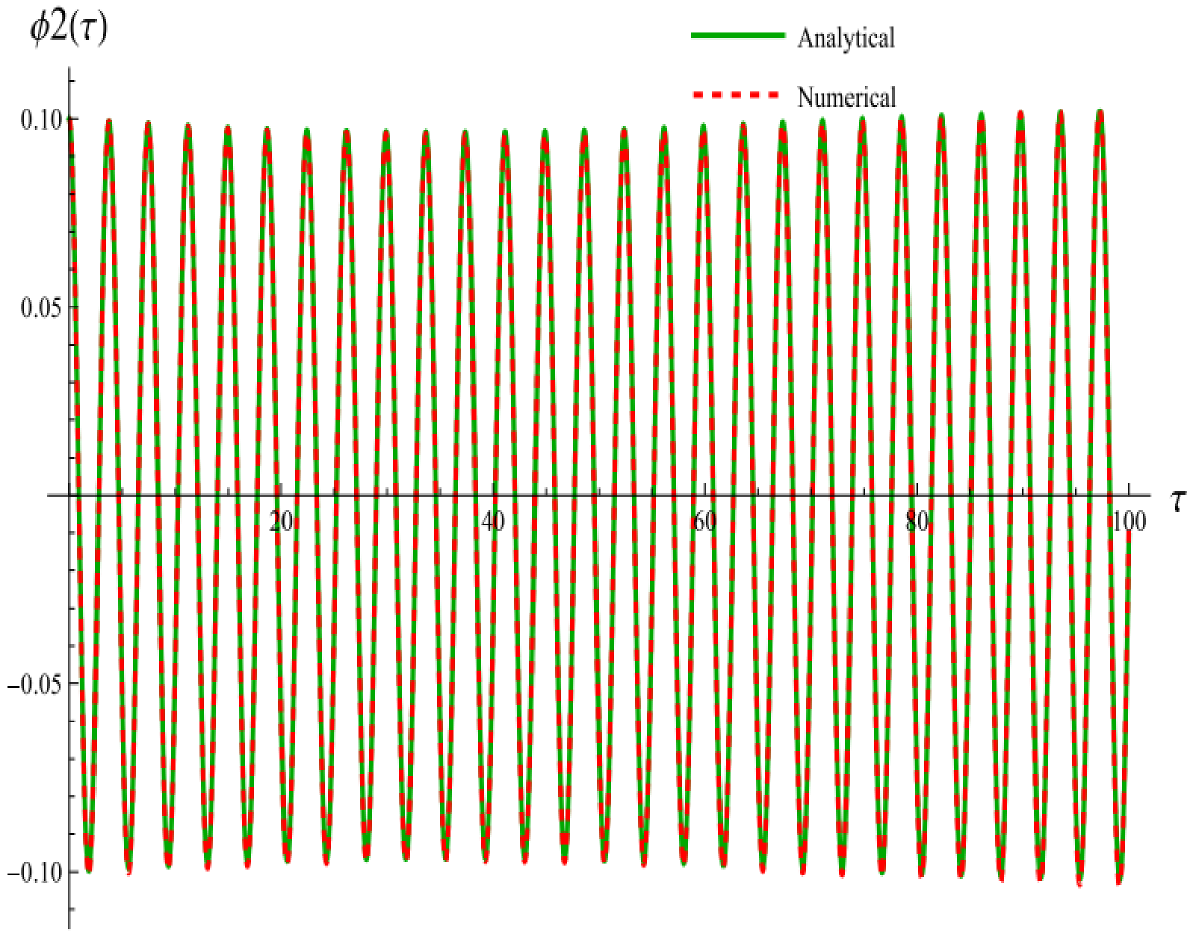


Fig.4b Comparison between the analytical and numerical solution of the 4-DOF modified SAM for $\phi_1(\tau)$ (iii), and $\phi_2(\tau)$ (iv)

Conclusions

- The Modified SAM presents a novel SAM concept applicable in the modeling of engineering objects.

- It is bases on a variable-length double pendulum with a suspension between the two pendulums.

- The derivation and form of the analytical solution obtained using the multiple scale method is complex. Although successful, some important simplifications had to be applied.

- Future work will be extended to resonance case and stability analysis.

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